

# Social Welfare in Budget Aggregation

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## Abstract

We study budget aggregation under  $\ell_1$ -utilities, a model for collective decision making in which agents with heterogeneous preferences must allocate a public budget across a set of alternatives. Each agent reports their preferred allocation, and a mechanism selects an allocation. Early work focused on social welfare maximization, which in this setting admits truthful mechanisms, but may underrepresent minority groups, motivating the study of *proportional* mechanisms. However, the dominant proportionality notion, *single-minded proportionality*, is weak, as it only constrains outcomes when agents hold extreme preferences. To better understand proportionality and its interaction with welfare and truthfulness, we address three questions.

First, how much welfare must be sacrificed to achieve proportionality? We formalize this via the *price of proportionality*, the best worst-case welfare ratio between a proportional mechanism and UTIL, the welfare-maximizing mechanism. We introduce a new single-minded proportional and truthful mechanism, UTILPROP, and show that it achieves the optimal worst-case ratio. Second, how do proportional mechanisms compare in terms of welfare? We define an instance-wise welfare dominance relation and use it to compare mechanisms from the literature. In particular, we show that UTILPROP welfare-dominates all previously known single-minded proportional and truthful mechanisms. Third, can stronger notions of proportionality be achieved without compromising welfare guarantees? We answer this question in the affirmative by studying *decomposability* and proposing GREEDYDECOMP, a decomposable mechanism with optimal worst-case welfare ratio. We further show that computing the welfare-dominant decomposable mechanism, UTILDECOMP, is NP-hard, and that GREEDYDECOMP provides a 2-approximation to UTILDECOMP in terms of welfare.

# 1 Introduction

Participatory budgeting, public investment planning, and resource allocation within organizations share a common challenge: agents with typically conflicting preferences must collectively decide how to allocate a public budget. In such settings, well-defined decision mechanisms are essential for transparency and legitimacy. Designing an appropriate mechanism, however, is nontrivial, as several desiderata must be balanced. Ideally, a mechanism should achieve high overall agreement (efficiency), ensure meaningful representation of all agents (proportionality), and incentivize truthful preference reporting (truthfulness). These goals, however, are often in tension.

We consider the model of budget aggregation under  $\ell_1$ -utilities Lindner et al. [2008], Goel et al. [2019]. Agents report their preferred budget allocations, and an agent’s utility for an outcome is modeled by the amount of budget spent in their interest; equivalently, disutility is measured by the  $\ell_1$ -distance between the outcome and the reported allocation. Unlike many voting settings in which truthfulness is unattainable, budget aggregation under  $\ell_1$ -utilities admits truthful mechanisms. In particular, Freeman et al. [2021] introduce a broad class of truthful mechanisms known as moving-phantom mechanisms. Earlier work on this model Lindner et al. [2008], Goel et al. [2019] focuses on maximizing social welfare, a goal that is compatible with truthfulness. In particular, Freeman et al. [2021] define UTIL as a neutral, anonymous, and truthful welfare-maximizing mechanism. They further characterize UTIL as the unique Pareto-optimal moving-phantom mechanism. A common concern with pure welfare maximization, however, is that it may disregard minority opinions.

To address this concern, Freeman et al. [2021] propose the INDEPENDENTMARKETS mechanism, which satisfies *single-minded proportionality*. This proportionality notion requires that when voters are *single-minded*—seeking to allocate the entire budget to a single alternative—the budget is split proportionally to the sizes of the corresponding supporter groups. Since single-minded proportionality is relatively weak, there exists a large class of truthful mechanisms satisfying it, motivating subsequent work that seeks to further distinguish among them, for example by bounding the distance to the *average* report Caragiannis et al. [2024], Freeman and Schmidt-Kraepelin [2024].

In this work, we pursue two orthogonal approaches to refine the comparison of single-minded proportional mechanisms and to reconnect the study of proportional mechanisms with the ideal of welfare maximization. First, we evaluate these mechanisms through the lens of welfare guarantees, which naturally leads to the study of the *price of proportionality*, that is, the best worst-case welfare ratio achievable by a single-minded proportional mechanism. Related questions have been studied for other utility models, for example by Michorzewski et al. [2020]. Beyond worst-case analysis, we also introduce a welfare dominance relation that enables direct welfare comparisons between mechanisms. Second, we propose *decomposability*, a notion of proportionality that is substantially stronger than single-minded proportionality in the sense that it constrains outcomes on all profiles rather than only single-minded ones. Overall, we present a systematic study of welfare, proportionality, and truthfulness in budget aggregation, uncovering two new mechanisms that strike a particularly favorable balance between welfare guarantees and proportionality.

Our first mechanism, UTILPROP, achieves the optimal worst-case approximation of social welfare among all single-minded proportional mechanisms, with welfare ratio  $\frac{n}{2\sqrt{n-1}}$  and  $\frac{m}{2\sqrt{m-2}}$ , where  $n$  is the number of voters and  $m$  is the number of alternatives. As it is a moving-phantom mechanism, UTILPROP is truthful. In addition to the worst-case guarantee, UTILPROP maximizes welfare in every instance among all single-minded proportional moving-phantom mechanisms, and hence among all known proportional and truthful mechanisms. To establish this result, we introduce a dominance relation: mechanism  $A$  dominates mechanism  $B$  if, for every instance,  $A$  returns an allocation with

welfare at least as high as that of  $B$ . This induces a partial order over moving-phantom mechanisms and allows us to characterize the relationships among those studied in the literature. For example, perhaps surprisingly, PIECEWISEUNIFORM Caragiannis et al. [2024] and LADDER Freeman and Schmidt-Kraepelin [2024] welfare-dominate the more established INDEPENDENTMARKETS mechanism. We refer to Figure 5 in Section 4 for an overview.

To strengthen single-minded proportionality, we draw inspiration from work on dichotomous preferences, where *decomposability* has been proposed as a proportionality notion Brandl et al. [2021]. Decomposability requires the total budget to be partitioned into  $n$  agent-specific allocations, each spent only on projects approved by the corresponding agent. We adapt this idea to the  $\ell_1$ -disutility model by requiring that the aggregate allocation can be decomposed into  $n$  agent-specific pieces of size  $1/n$ , such that each agent contributes only to alternatives whose funding level does not exceed that agent’s preferred level. Equivalently, no agent would prefer to reallocate their share of the budget given the contributions of the others. This condition mirrors a Nash equilibrium in an associated budget-allocation game and coincides with the *equilibrium* notion introduced recently and independently by Becker et al. [2025].

We show, however, that decomposability is incompatible with truthfulness and neutrality, and therefore drop truthfulness in this part of the paper. We introduce a new decomposable mechanism, GREEDYDECOMP, which—like UTILPROP—minimizes worst-case welfare loss (with respect to  $n$ ) within the class of single-minded proportional mechanisms. In particular, this demonstrates that strengthening proportionality beyond the single-minded case does not increase the price of proportionality. Although GREEDYDECOMP is worst-case optimal with respect to welfare loss, it is not optimal on every instance among decomposable mechanisms. In fact, we show that computing the outcome of UTILDECOMP, the mechanism that maximizes welfare among all decomposable mechanisms, is NP-hard. On the positive side, we prove that GREEDYDECOMP approximates the welfare of UTILDECOMP with a factor of 2. We illustrate our approximation results in Figure 1.

## 1.1 Related Work

**Budget Aggregation under  $\ell_1$ -Utilities** We contribute to the growing literature on budget aggregation under  $\ell_1$ -utilities. As mentioned above, Lindner et al. [2008] and Goel et al. [2019] initiate the study of welfare-maximizing mechanisms and Freeman et al. [2021] introduce the class of moving-phantom mechanisms, including UTIL. As noted before, UTIL is not single-minded proportional, and in fact, Brandt et al. [2024] prove that no mechanism can simultaneously be truthful, Pareto optimal, and single-minded proportional, thereby strengthening an impossibility for the class of moving-phantom mechanisms by Freeman et al. [2021]. One may wonder whether there exist compelling truthful mechanisms outside the class of moving-phantom mechanisms. Indeed, de Berg et al. [2024] demonstrate the existence of other truthful mechanisms that are also anonymous and neutral. However, none of these mechanisms satisfies single-minded proportionality, hence, they are

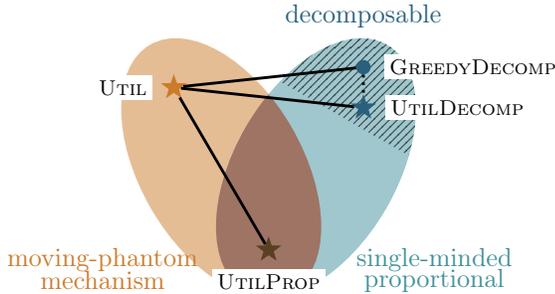


Figure 1: Mechanisms that are instance-wise welfare-optimal within their respective class are marked with stars. The solid lines correspond to a worst-case welfare loss of  $\frac{n}{2\sqrt{n-1}}$  compared to UTIL. The dotted line indicates a welfare loss of at most  $2 - \frac{1}{n-1}$ .

not suitable for our purposes. In terms of fairness notions that go beyond single-minded proportionality, Caragiannis et al. [2024] introduce the idea of mean approximation for the budget aggregation problem, which has later been followed up by Freeman and Schmidt-Kraepelin [2024]. These two papers identify the two proportional moving-phantom mechanisms PIECEWISEUNIFORM and LADDER as particularly well-performing in terms of mean approximation. Recently, Becker et al. [2025] introduced budget-aggregation games, in which agents control fractions of a public budget and allocate their shares to maximize their utility. They show that for  $\ell_1$ -utilities, Nash equilibria can be computed in polynomial time. While these equilibria correspond to our fairness notion of decomposability, their proposed algorithm is not neutral. In contrast, our algorithm GREEDYDECOMP is polynomial-time computable, neutral, anonymous, and worst-case welfare-optimal. Elkind et al. [2026] compare budget-aggregation mechanisms with respect to a broad range of axioms, including both inter-profile and fairness axioms. Goyal et al. [2023] study distortion and sample complexity in a setting similar to ours, except that each alternative’s funding is capped by a predefined cost.

**Price of Proportionality in Social Choice** While the price of proportionality [Bertsimas et al., 2011, Caragiannis et al., 2012] has not been studied in our model, we discuss similar results in related settings. In budget division under approval preference, Michorzewski et al. [2020] show that asking for the property of *individual fair share (IFS)* (which is incomparable to single-minded proportionality in our setting) necessarily yields a worst-case welfare ratio of  $\frac{\sqrt{m}}{2}$ , where  $m$  is the number of projects, and show that three common algorithms achieve a worst-case welfare ratio between  $\frac{\sqrt{m}}{2}$  and  $\frac{m}{2\sqrt{m-2}}$ .<sup>1</sup> While we express most of our bounds in terms of  $n$ , the lower bound  $\frac{\sqrt{m}}{2}$  applies for us well, and in Section 3.4 we combine our techniques with those of Michorzewski et al. [2020] to show that UTILPROP also achieves a  $\frac{m}{2\sqrt{m-2}}$  approximation, thereby providing an alternative (almost) tight bound on the price of single-minded proportionality, expressed in terms of  $m$ . Tang et al. [2020] study the price of fairness with respect to egalitarian welfare.

In approval-based committee voting, the study of the trade-off between proportionality and utilitarian welfare was initiated by Lackner and Skowron [2020], who, among other results, showed that any rule satisfying the property of *justified representation* must have a welfare approximation ratio of at least  $\frac{k}{2\sqrt{k-1}}$ , where  $k$  is the committee size. Elkind et al. [2024] later demonstrated that this lower bound is asymptotically tight by providing a matching upper bound. Brill and Peters [2024] further improved this result by showing that an approximation of  $\frac{k}{2\sqrt{k-1}}$  can be achieved by completing the *Greedy Justified Candidate Rule* Brill and Peters [2023].

**Outline** In Section 2, we introduce our model and moving-phantom mechanisms. Section 3 defines the proportionality notions of single-minded proportionality and decomposability, and presents the algorithms UTILPROP and GREEDYDECOMP. To establish their optimal welfare guarantees, we first derive a lower bound and then prove a meta-theorem showing that any mechanism satisfying range respect Elkind et al. [2026] and a newly defined property called *proportional spending* achieves worst-case optimal welfare among all single-minded proportional mechanisms. Section 4 introduces our welfare dominance relation and characterizes the welfare ordering among eight moving-phantom mechanisms. In Section 5, we show that computing UTILDECOMP is NP-hard, while GREEDYDECOMP achieves a 2-approximation in welfare. Finally, Section 6 concludes with a discussion of relevant extensions and directions for future work.

<sup>1</sup>To allow comparisons, we invert the bounds from the literature, which are reported as efficiency guarantees  $\min(\text{ALG}/\text{OPT})$ , where ALG is the mechanism’s welfare and OPT is the welfare of UTIL. Our objective instead is  $\max(\text{OPT}/\text{ALG})$ .

## 2 Preliminaries

For  $n \in \mathbb{N}$ , we let  $[n] = \{k \in \mathbb{N} \mid 1 \leq k \leq n\}$  and  $[n]_0 = \{k \in \mathbb{N} \mid 0 \leq k \leq n\}$ . For  $k \in \mathbb{N}$ , we write  $\Delta^{(k)} = \{v \in [0, 1]^{k+1} \mid \sum_{k'=1}^{k+1} v_{k'} = 1\}$  for the standard  $k$ -simplex.

**Budget Aggregation** In budget aggregation,  $n$  voters want to decide how to distribute a normalized budget of 1 over  $m \geq 2$  alternatives. Each voter  $i \in [n]$  reports their favored allocation  $p_i \in \Delta^{(m-1)}$  as their *vote*, and all votes together form the *profile*  $P = (p_1, \dots, p_n)$ . We let  $\mathcal{P}_{n,m} = (\Delta^{(m-1)})^n$  denote the set of all possible profiles with  $n$  voters and  $m$  alternatives. For a profile  $P \in \mathcal{P}_{n,m}$  and  $k \in [n]$ , let the vector  $\mu^k \in [0, 1]^m$  be defined by  $\mu_j^k$  being the  $k$ -th smallest element among  $\{p_{i,j} \mid i \in [n]\}$  for each  $j \in [m]$ . Thus,  $\mu^k$  generally does not correspond to the vote of any single agent; for instance,  $\mu^1$  corresponds to the vector of minimum votes per alternative.

A *budget-aggregation mechanism* (often simply called *mechanism*) is a family of functions  $\mathcal{A}_{n,m}: \mathcal{P}_{n,m} \rightarrow \Delta^{(m-1)}$ , one for each pair  $n, m \in \mathbb{N}$  with  $m \geq 2$ , that each map a profile  $P \in \mathcal{P}_{n,m}$  to an *aggregate*  $a = \mathcal{A}_{n,m}(P) \in \Delta^{(m-1)}$ . When  $n$  and  $m$  are clear from the context, we frequently write  $\mathcal{A}$  instead of  $\mathcal{A}_{n,m}$ .

**Utilities and Social Welfare** We define the *overlap* of two vectors  $v, w \in \mathbb{R}^m$  to be  $\sqcap(v, w) = \sum_{j \in [m]} \min(v_j, w_j)$  and the utility of a voter  $i$  with preference  $p_i \in \Delta^{(m-1)}$  for an allocation  $a \in \Delta^{(m-1)}$  as  $u_i(a) = \sqcap(p_i, a)$ . This utility function has been studied previously [Goel et al., 2019, Nehring and Puppe, 2019] and induces the same ordinal rankings over budget allocations as  $\ell_1$ -disutilities (i.e., the  $\ell_1$  distance between the vote and the aggregate). However, our results on welfare approximations depend on the specific utility formulation. Later in the paper, we occasionally use the informal expression that a voter  $i$  *approves* increasing an allocation from  $a_j$  to  $a'_j > a_j$ . This is merely shorthand for the condition  $p_{i,j} \geq a'_j$ . We denote by  $w(a) = \sum_{i \in [n]} u_i(a)$  the (utilitarian) social welfare of an allocation  $a$ ; we often refer to this value simply as the *welfare* of an allocation, for conciseness. For a fixed profile  $P \in \mathcal{P}_{n,m}$ , we let  $a^{\text{OPT}} \in \arg \max\{w(a) \mid a \in \Delta^{(m-1)}\}$  denote a welfare-optimal allocation and  $w^{\text{OPT}} = w(a^{\text{OPT}})$  denote the corresponding optimal welfare. For  $n \in \mathbb{N}$  and some function  $\alpha: \mathbb{N} \rightarrow \mathbb{R}_{\geq 1}$ , we say that a mechanism  $\mathcal{A}$  achieves a welfare approximation of  $\alpha$  if, for any  $m \in \mathbb{N}$  and instance  $P \in \mathcal{P}_{n,m}$ , it holds that  $\frac{w^{\text{OPT}}}{w(\mathcal{A}(P))} \leq \alpha(n)$ .

**Axioms** All mechanisms considered in this paper satisfy two basic symmetry notions with respect to voters and candidates. A mechanism is *anonymous* if its output is invariant under permutations of the voters and *neutral* if a permutation of the alternatives in all individual reports permutes the outcome in the same way.

For a profile  $P \in \mathcal{P}_{n,m}$ , we say that an allocation  $a$  *Pareto dominates* another allocation  $a'$  if  $u_i(a) \geq u_i(a')$  for all voters  $i \in [n]$  and  $u_i(a) > u_i(a')$  for at least one voter  $i \in [n]$ . We call an allocation *Pareto optimal* if it is *not* Pareto dominated by any other allocation.

A budget-aggregation mechanism  $\mathcal{A}$  is *truthful* if, for all  $n, m \in \mathbb{N}$  with  $m \geq 2$ , profile  $P = (p_1, \dots, p_n) \in \mathcal{P}_{n,m}$ , voter  $i \in [n]$ , and misreport  $p'_i \in \Delta^{(m-1)}$ , we have  $u_i(\mathcal{A}(P)) \geq u_i(\mathcal{A}(p'_i, P_{-i}))$ , where  $(p'_i, P_{-i})$  is a shorthand for  $(p_1, \dots, p_{i-1}, p'_i, p_{i+1}, \dots, p_n)$ . Freeman et al. [2021] introduced a class of truthful mechanisms, which we define next.

**Moving-Phantom Mechanisms.** A moving-phantom mechanism is defined by one *phantom system* for each number of voters. For  $n \in \mathbb{N}$ , a *phantom system*  $\mathcal{F}_n = \{f_k \mid k = 0, \dots, n\}$  is a

collection of  $(n + 1)$  non-decreasing, continuous functions  $f_k: [0, 1] \rightarrow [0, 1]$  with  $f_k(0) = 0$ ,  $f_k(1) \geq 1 - \frac{k}{n}$  and  $f_0(t) \geq \dots \geq f_n(t)$  for all  $t \in [0, 1]$ . A family of phantom systems  $\mathcal{F} = \{\mathcal{F}_n \mid n \in \mathbb{N}\}$  defines the moving-phantom mechanism  $\mathcal{A}^{\mathcal{F}}$  in the following way. For any given profile  $P = (p_1, \dots, p_n)$  with  $p_i = (p_{i,1}, \dots, p_{i,m})$  for each  $i \in [n]$ , let  $t^* \in [0, 1]$  be a value for which

$$\sum_{j \in [m]} \text{med}(f_0(t^*), \dots, f_n(t^*), p_{1,j}, \dots, p_{n,j}) = 1.$$

Note that the median is well-defined since its input consists of an odd number of values. The moving-phantom mechanism  $\mathcal{A}^{\mathcal{F}}$  induced by  $\mathcal{F}$  returns the allocation  $a = (a_1, \dots, a_m)$ , where  $a_j = \text{med}(f_0(t^*), \dots, f_n(t^*), p_{1,j}, \dots, p_{n,j})$  for each  $j \in [m]$ . Although  $t^*$  is not always unique, the induced allocation is, and we refer to  $t^*$  as a *time of normalization*. Freeman et al. [2021] showed that any moving-phantom mechanism is truthful, anonymous, and neutral.

In Appendix A, we provide formal definitions and visualizations of all moving-phantom mechanisms introduced in previous literature and in this paper; we here give an intuitive description of those that are most relevant for our work. The UTIL mechanism [Freeman et al., 2021] is induced by moving each phantom from 0 to 1 consecutively, one after each other. INDEPENDENTMARKETS [Freeman et al., 2021] is induced by moving all phantoms simultaneously but at different speeds, namely, phantom  $k$  moves with speed  $\frac{n-k}{n}$ . The PIECEWISEUNIFORM mechanism [Caragiannis et al., 2024] is induced by a two-phase moving procedure: a first phase that spreads the uppermost half of the phantoms uniformly across the  $[0, 1]$  interval while the lower half remains at zero, followed by a second phase that spreads the lower half of the phantoms across  $[0, \frac{1}{2}]$  while concentrating the upper half of the phantoms into  $[\frac{1}{2}, 1]$ . The LADDER mechanism [Freeman and Schmidt-Kraepelin, 2024] is induced by moving each phantom at the same speed, but with phantom  $k$  starting to move only when phantom  $k - 1$  reaches position  $\frac{1}{n}$ . Finally, GREEDYMAX [de Berg et al., 2024] is induced by moving the top  $n$  phantoms together from 0 to 1, while the last one remains at 0.

### 3 Price of Proportionality

In this section, we define two notions of proportionality and discuss the extent to which social welfare must be sacrificed to satisfy them. We quantify this welfare loss via the best welfare approximation function  $\alpha(n)$  that a mechanism satisfying a certain proportionality notion can guarantee, which we refer to as the *price* of the corresponding proportionality notion. After defining single-minded proportionality and decomposability in Section 3.1, we present mechanisms satisfying them in Section 3.2, and analyze their welfare guarantees in Sections 3.3 and 3.4.

#### 3.1 Proportionality Notions

**Single-minded proportionality** Although there is no single “correct” definition of proportionality in budget aggregation, there are some preference profiles for which proportionality is arguably uncontroversial. In particular, when each voter wishes to allocate the entire budget to a single alternative, a proportional allocation assigns to each alternative a budget share equal to its fraction of supporters. Freeman et al. [2021] formalize this intuition with the property of *single-minded proportionality*.<sup>2</sup> A vote  $p_i \in \Delta^{m-1}$  is called *single-minded* if there is an alternative  $j \in [m]$  with  $p_{i,j} = 1$

<sup>2</sup>While Freeman et al. [2021] call this property *proportionality*, we reserve the term proportionality as an umbrella term.

and a profile  $P \in \mathcal{P}_{n,m}$  is called single-minded if all votes are single-minded. A budget aggregation mechanism  $\mathcal{A}$  is *single-minded proportional* if, for any single-minded profile  $P = (p_1, \dots, p_n) \in \mathcal{P}_{n,m}$ , it returns the average of the votes, i.e.,  $\mathcal{A}(P) = \frac{1}{n} \sum_{i \in [n]} p_i$ .

Several mechanisms in the literature are single-minded proportional and truthful. Freeman et al. [2021] introduce the INDEPENDENTMARKETS mechanism; Caragiannis et al. [2024] and Freeman and Schmidt-Kraepelin [2024] introduce the PIECEWISEUNIFORM and LADDER mechanisms, respectively, and bound their distance from the average allocation. In Section 3.2, we introduce UTILPROP, which achieves best-possible welfare approximation among single-minded proportional mechanisms.

**Decomposability** The main issue with single-minded proportionality is that it only restricts the outcome of a mechanism for very specific profiles. To obtain a stronger proportionality notion, we draw inspiration from the concept of *decomposability*, defined by Brandl et al. [2021] for the approval preference setting. It requires that the outcome can be decomposed into voter-specific partial allocations, called contributions, each summing to  $\frac{1}{n}$ , such that every voter is satisfied with their contribution. In the approval setting, this simply means that voters contribute only to alternatives that they approve. In our setting, however, satisfaction depends on the contributions of others. We interpret a voter as satisfied if they do not fund an alternative that receives more total funding than they voted for. Otherwise, the voter would prefer to redirect that portion of their contribution to an alternative that is currently underfunded from their perspective. Such an alternative must exist by normalization of the votes and the outcome.

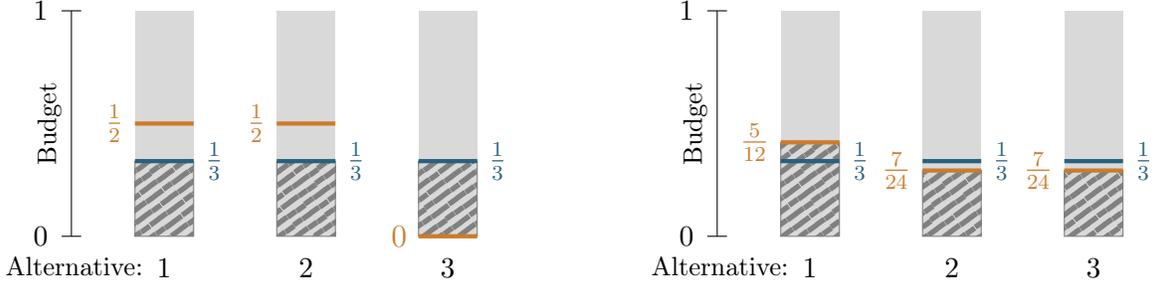
Formally, an aggregate  $a$  is *decomposable* for a profile  $P$  if there exist voter contributions  $c_i \in [0, \frac{1}{n}]^m$  with  $\sum_{j \in [m]} c_{i,j} = \frac{1}{n}$  for each voter  $i \in [n]$ , such that

- $\sum_{i \in [n]} c_{i,j} = a_j$  for each alternative  $j \in [m]$ , and
- $a_j \leq p_{i,j}$  for all voters  $i$  and alternatives  $j$  with  $c_{i,j} > 0$ .

A mechanism  $\mathcal{A}$  is called decomposable if it returns a decomposable aggregate for every profile. Another way to think about decomposability is as a Nash equilibrium of the game in which each voter can distribute their fair share as they like. Becker et al. [2025] call these *budget-aggregation games* and show that such an equilibrium always exists and is computable in polynomial time, implying the existence of a polynomial-time decomposable mechanism. In Section 3.2, we present a different polynomial-time decomposable mechanism, GREEDYDECOMP, that achieves the best-possible welfare approximation, even within the class of single-minded proportional mechanisms. Note that decomposability implies single-minded proportionality, since any single-minded voter can only contribute to their supported alternative. However, when restricting to mechanisms satisfying our baseline property of neutrality, this strengthening comes at the cost of truthfulness.

**Proposition 1.** *No neutral and decomposable budget aggregation mechanism is truthful.*

*Proof.* Consider the profile  $P \in \mathcal{P}_{2,3}$  with  $p_1 = (\frac{1}{2}, \frac{1}{2}, 0)$  and  $p_2 = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$  (see Figure 2a). Let  $a$  be the output of a neutral, decomposable mechanism for  $P$  and let  $c_1$  and  $c_2$  be voter contributions certifying decomposability of  $a$ . We claim that  $a = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ . Note that, by neutrality, we have  $a_1 = a_2$ . Suppose first that  $a_1 = a_2 > \frac{1}{3}$ . Then, voter 2 can not contribute to alternative 1 and 2, i.e.,  $c_{2,1} = c_{2,2} = 0$ , which implies that this voter must put their whole contribution on alternative 3, i.e.,  $c_{2,3} = \frac{1}{2}$ . But then, we have  $a_3 \geq \frac{1}{2} > \frac{1}{3} = p_{2,3}$ , contradicting decomposability. On the other hand, if  $a_1 = a_2 < \frac{1}{3}$ , then we have  $a_3 > \frac{1}{3} = p_{2,3} > p_{1,3}$ , contradicting decomposability.



(a) Profile  $P$ . The aggregate of any neutral and decomposable mechanism is  $a = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ .

(b) Profile  $P'$ . The aggregate of any neutral and decomposable mechanism is  $a' = (\frac{5}{12}, \frac{7}{24}, \frac{7}{24})$ .

Figure 2: Visualization of the profiles from the proof of Proposition 1. Voters are drawn as horizontal (blue and orange) lines and aggregates are drawn as (gray) hatched areas.

Now, consider the profile  $P' \in \mathcal{P}_{2,3}$  in which voter 1 misreports  $p'_1 = (\frac{5}{12}, \frac{7}{24}, \frac{7}{24})$  (see Figure 2b). Let  $a'$  be the output of a neutral, decomposable mechanism for  $P'$ , and let  $c'_1$  and  $c'_2$  be voter contributions certifying decomposability of  $a'$ . We claim that  $a' = (\frac{5}{12}, \frac{7}{24}, \frac{7}{24})$ . By neutrality,  $a'_2 = a'_3$ . Suppose first that  $a'_2 = a'_3 > \frac{7}{24}$ . Then, voter 1 cannot contribute to alternative 2 and 3, i.e.,  $c_{1,2} = c_{1,3} = 0$ , which implies that this voter must put their whole contribution on alternative 1, i.e.,  $c_{1,1} = \frac{1}{2}$ . But then, we have  $a'_1 \geq \frac{1}{2} > \frac{5}{12} = p'_{1,1}$ , contradicting decomposability. On the other hand if  $a'_2 = a'_3 < \frac{7}{24}$ , then we have  $a'_1 > \frac{5}{12} = p'_{1,1} > p_{2,1}$ , contradicting decomposability.

Since voter 1 with preference  $p_1$  profits from misreporting, i.e.,  $u_1(a) = \frac{2}{3} < \frac{17}{24} = u_1(a')$ , we conclude from these profiles that any neutral and decomposable mechanism is not truthful.  $\square$

### 3.2 Proportional Mechanisms

We now introduce two single-minded proportional mechanisms, UTILPROP and GREEDYDECOMP, where UTILPROP is additionally truthful and GREEDYDECOMP satisfies decomposability.

**UTILPROP** The UTILPROP mechanism is the moving-phantom mechanism induced by

$$f_k(t) = \max \left( 0, \min \left( \frac{n-k}{n}, t(n+1) - k \right) \right),$$

which corresponds to moving each phantom with index  $k$  from 0 to  $\frac{n-k}{n}$  consecutively.

Freeman et al. [2021] show that, if there is a time such that each phantom with index  $k$  is at position  $\frac{n-k}{n}$ , then the corresponding mechanism is single-minded proportional. We will show in Lemma 5 that in fact this condition is also necessary. For now, Proposition 2 is an immediate implication of their result (for time  $t = 1$ ) and the fact that UTILPROP is a moving-phantom mechanism.

**Proposition 2.** *UTILPROP is truthful and single-minded proportional.*

In addition to truthfulness, UTILPROP satisfies neutrality and anonymity, since it is a moving-phantom mechanism.

**GREEDYDECOMP** The underlying idea of GREEDYDECOMP is to iteratively allocate the budget to alternatives where the maximum number of voters approve of the spending. However, since we aim for decomposability, we need to ensure that any marginal spending is covered by the budget of a voter above it. To do this, we always charge spending to the highest-voting voter with a positive remaining budget. This choice is “safe”, as it avoids later increasing the spending on some alternative above the vote of a voter who has already contributed to that alternative, which would violate decomposability.

As a warm-up, we describe a non-neutral, non-anonymous version of this mechanism. The algorithm starts with an empty allocation  $\tilde{a} = (0, \dots, 0)$  and voter budgets  $b_i = \frac{1}{n}$  for every voter  $i$ , and it spends these budgets over  $n$  iterations. In each iteration  $k$ , we go through the alternatives in some arbitrary order. For each alternative, we aim to increase  $\tilde{a}_j$  to  $\mu_j^k$  (the  $k^{\text{th}}$  lowest vote on alternative  $j$ ), if the (at least  $n + 1 - k$ ) voters approving this marginal increase can afford it with their own budget. We first check if there are voters with a positive budget who vote higher than  $\tilde{a}_j$ . If not, we skip this alternative. Otherwise, we find the highest-voting voter  $i$  with positive budget  $b_i > 0$  (subject to some tie-breaking) and increase  $\tilde{a}_j$  while charging voter  $i$  for it. We do this until either voter  $i$  runs out of budget, in which case we look for the next highest vote above  $\tilde{a}_j$ , or until  $\tilde{a}_j$  reaches  $\mu_j^k$ , in which case we move on to the next alternative. Note that if  $\tilde{a}_j$  reaches the vote of voter  $i$  (the current maximum voter with positive budget), then  $\tilde{a}_j$  also reaches  $\mu_j^k = p_{i,j}$ . Thus, there is no voter with positive budget voting higher than  $\tilde{a}_j$  anymore and we will not spend any more budget on alternative  $j$  in further iterations, which is important for decomposability. This has the effect that in each iteration  $k$  the algorithm will only consider increasing  $\tilde{a}_j$  to  $\mu_j^k$ , if we already spent  $\tilde{a}_j = \mu_j^{k-1}$  (where we denote  $\mu_j^0 = 0$ ) on alternative  $j$ .

While the procedure described above is already a decomposable mechanism, we formalize an anonymous and neutral version of it as GREEDYDECOMP in Algorithm 1. The conceptual differences are that, in each iteration  $k$ , (1) we spend the budget on all alternatives at the same time and (2) in case there are multiple voters with the maximum vote, we charge all of them equally when increasing  $\tilde{a}$ . To achieve this, in each iteration  $k$ , we first identify for every alternative  $j$  the set  $N_j^+$  of voters with the maximum vote on  $j$ , among those with votes higher than  $\tilde{a}_j$  that still have a positive budget. Given an alternative  $j$  and some value  $\tau \in [0, 1]$ , we charge each voter  $i \in N_j^+$  a payment of  $\pi_{i,j}(\tau) = \max\left(0, \frac{\min(\mu_j^k, \tau) - \tilde{a}_j}{|N_j^+|}\right)$  and the other voters  $i \in [n] \setminus N_j^+$  pay  $\pi_{i,j}(\tau) = 0$ . We can think of the value  $\tau$  as a proposal to increase  $\tilde{a}$  on each alternative  $j$  to  $\tau$  whenever  $\tau$  lies between  $\tilde{a}_j$  and  $\mu_j^k$ , that is, to  $\max(\tilde{a}_j, \min(\mu_j^k, \tau))$ . Each proposed value  $\tau$  then comes with payments  $\pi_{i,j}$  for each voter  $i \in N_j^+$  on each alternative  $j$ , and we need to make sure that no voter pays more than they have. Therefore, we compute the maximum value  $\tau^* \leq 1$ , such that  $b_i \geq \sum_{j \in [m]} \pi_{i,j}(\tau^*)$  for all voters  $i \in [n]$ . For each alternative  $j \in [m]$ , we then update  $\tilde{a}_j$  to  $\tilde{a}_j + \sum_{i \in [n]} \pi_{i,j}(\tau^*)$  and  $b_i$  to  $b_i - \pi_{i,j}(\tau^*)$  for  $i \in N_j^+$ . We repeat this iteratively, until  $\tau^* = 1$ , and then continue with the next iteration for  $k$ . Note that, for all alternatives with  $N_j^+ \neq \emptyset$ , the update of  $\tilde{a}_j$  can equivalently be written as  $\tilde{a}_j = \max(\tilde{a}_j, \min(\mu_j^k, \tau^*))$ . We thus never increase  $\tilde{a}_j$  above  $\mu_j^k$  (and therefore never above  $p_{i,j}$  for any  $i \in N_j^+$ ) in iteration  $k$  and never increase  $\tilde{a}_j$  at all if there are no voters with positive budget voting above it, i.e.,  $N_j^+ = \emptyset$ .

The following proposition, proven in Appendix B.1, summarizes the main properties of GREEDYDECOMP: It outputs a feasible aggregate and it satisfies decomposability.

**Proposition 3.** *GREEDYDECOMP is a decomposable budget aggregation mechanism and can be computed in polynomial time.*

---

**ALGORITHM 1: GREEDYDECOMP**

---

```
1  $\tilde{a} \leftarrow (0)_{j \in [m]}$ ;
2  $b_i \leftarrow \frac{1}{n}$  for each  $i \in [n]$ ;
3 for  $k = 1, \dots, n$  do // “outer iterations”
4    $\tau^* = 0$ ;
5   while  $\tau^* \neq 1$  do // “inner iterations”
6      $N_j^+ \leftarrow \arg \max \{p_{i,j} : i \in [n] \text{ s.t. } b_i > 0 \text{ and } p_{i,j} > \tilde{a}_j\}$  for each  $j \in [m]$ ;
7     choose  $\tau^* \leq 1$  maximal such that  $b_i \geq \sum_{j \in [m]} \pi_{i,j}(\tau^*)$  for all voters  $i \in [n]$ , where
       $\pi_{i,j}(\tau^*) = \max\left(0, \frac{\min(\mu_j^k, \tau^*) - \tilde{a}_j}{|N_j^+|}\right)$  if  $i \in N_j^+$  and  $\pi_{i,j}(\tau^*) = 0$  otherwise;
8     for  $j \in [m], i \in [n]$  do
9        $b_i \leftarrow b_i - \pi_{i,j}(\tau^*)$ ;
10       $\tilde{a}_j \leftarrow \tilde{a}_j + \pi_{i,j}(\tau^*)$ ;
11    end
12  end
13 end
```

---

### 3.3 Welfare Approximation

Given the existence of single-minded proportional and decomposable mechanisms, we now turn to the question of how much welfare has to be sacrificed to satisfy these properties. The function  $\alpha^*: \mathbb{N} \rightarrow \mathbb{R}$ , defined as

$$\alpha^*(n) = \max_{\ell \in [n]} \frac{n\ell}{n + \ell(\ell - 1)} \quad \text{for all } n \in \mathbb{N},$$

plays a key role in what follows. The following simple proposition, whose proof is deferred to Appendix B.2, provides a simple upper bound on the value of this function for each  $n$ .

**Proposition 4.** *For every  $n \in \mathbb{N}$ , we have  $\alpha^*(n) \leq \frac{n}{2\sqrt{n-1}}$ .*

We first show that  $\alpha^*(n)$  provides a lower bound on the welfare approximation achievable by any single-minded proportional mechanism. We note that a closely related instance was used by Michorzewski et al. [2020, Theorem 1] to show that, under approval preferences, the welfare loss implied by individual fair share is at least  $\frac{\sqrt{m}}{2}$ . Since their construction is single-minded and thus falls within our model, and since individual fair share is weaker than single-minded proportionality on single-minded instances, the same lower bound of  $\frac{\sqrt{m}}{2}$  carries over to our setting.

**Proposition 5.** *For every  $n \in \mathbb{N}$  and every single-minded proportional budget aggregation mechanism  $\mathcal{A}$ , there exists  $m \in \mathbb{N}$  and a profile  $P \in \mathcal{P}_{n,m}$  such that  $\frac{w^{\text{OPT}}}{w(\mathcal{A}(P))} \geq \alpha^*(n)$ .*

*Proof.* Fix the values  $n \in \mathbb{N}$  and  $\ell \in [n]$  and the mechanism  $\mathcal{A}$ , and consider the following single-minded instance with  $n$  voters and  $m = n - \ell + 1$  alternatives. The first  $n - \ell$  voters each want to allocate their entire budget to a different alternative, i.e., for  $i \in [n - \ell]$  we have  $p_{i,j} = 1$  if  $i = j$  and  $p_{i,j} = 0$  otherwise. The remaining  $\ell$  voters want to allocate their entire budget to the last alternative, i.e., for  $i \in \{n - \ell + 1, \dots, n\}$  we have  $p_{i,j} = 1$  if  $j = m$  and  $p_{i,j} = 0$  otherwise. See Figure 3 for a visualization of this profile in the case where  $\ell = \sqrt{n} \in \mathbb{N}$ .

Since  $\mathcal{A}$  is single-minded proportional we have  $\mathcal{A}(P) = (\frac{1}{n}, \dots, \frac{1}{n}, \frac{\ell}{n})$ , while an optimal outcome in terms of welfare (in fact, the unique one when  $\ell > 1$ ) is  $a^{\text{OPT}} = (0, \dots, 0, 1)$ . Thus,

$$\frac{w(a^{\text{OPT}})}{w(\mathcal{A}(P))} = \frac{\ell}{\frac{1}{n}(n - \ell) + \frac{\ell}{n}\ell} = \frac{n\ell}{n + \ell(\ell - 1)}.$$

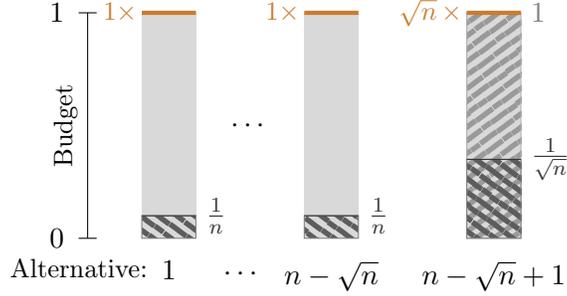


Figure 3: Example instance used for the proof of Proposition 5. Only non-zero votes are drawn. The numbers to the left of the votes (“ $\sqrt{n} \times$ ”) indicate the number of voters voting that value. The aggregate of any single-minded proportional mechanism is drawn in dark gray (downwards hatching) and the welfare-optimal aggregate in light gray (upwards hatching).

Since this analysis is valid for any  $\ell \in [n]$ , it is valid in particular for the one that maximizes the last term, which finishes the proof.  $\square$

In the remainder of this section, we show that the above lower bound on the welfare approximation that single-minded proportional mechanisms can guarantee is tight and can be attained by both UTILPROP and GREEDYDECOMP. This implies that, on top of a best worst-case welfare guarantee, we can either add truthfulness or strengthen single-minded proportionality to decomposability. To prove that both mechanisms meet the welfare bound from Proposition 5, we introduce two properties of budget aggregation mechanisms that, together, suffice to achieve this guarantee. We then show that UTILPROP and GREEDYDECOMP satisfy these properties.

**Range respect** The first property states that a mechanism should never spend less than the minimum vote or more than the maximum vote on each alternative. Formally, a mechanism  $\mathcal{A}$  is called *range respecting* if, for every profile  $P \in \mathcal{P}_{n,m}$ , the output of  $\mathcal{A}$  lies between the minimum and maximum vote on each alternative  $j$ , i.e.,  $\mu_j^1 = \min_{i \in [n]} p_{i,j} \leq \mathcal{A}(P)_j \leq \max_{i \in [n]} p_{i,j} = \mu_j^n$ .

**Proportional spending** The second property requires a mechanism to spend, for any  $k \in [n]$ , at least  $\frac{n-k+1}{n}$  in total below the ceiling formed by the  $k^{\text{th}}$  lowest votes on the alternatives (if possible), since that is the spending supported by  $n - k + 1$  voters. Formally, a mechanism satisfies *proportional spending* if, for every profile  $P \in \mathcal{P}_{n,m}$  and  $k \in [n]$ , we have

$$\square(\mathcal{A}(P), \mu^k) \geq \min \left( \frac{n - k + 1}{n}, \sum_{j \in [m]} \mu_j^k \right).$$

These two properties together guarantee the best-possible welfare approximation constrained to single-minded proportionality, as stated in the following theorem.

**Theorem 1.** *Any range-respecting mechanism satisfying proportional spending attains a welfare approximation of  $\alpha^*(n)$  for every  $n \in \mathbb{N}$ .*

To prove this theorem, we introduce one more lemma about the welfare-maximizing aggregate returned by UTIL. The lemma states that UTIL satisfies a stronger version of proportional spending:

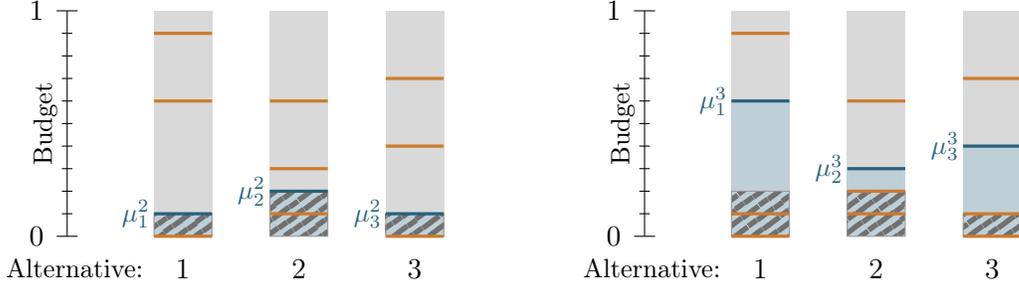


Figure 4: Illustration of the proportional spending property on an instance with  $n = 4$  voters and  $m = 3$  alternatives. For  $k = 2$  (left figure), we have  $\frac{4}{10} = \sum_{j \in [m]} \mu_j^k < \frac{n-k+1}{n} = \frac{3}{4}$ , thus proportional spending requires that  $a_j \geq \mu_j^k$  for each alternative  $j$ . For  $k = 3$  (right figure), we have  $\frac{13}{10} = \sum_{j \in [m]} \mu_j^k > \frac{n-k+1}{n} = \frac{1}{2}$ , thus proportional spending requires that  $a_j$  spends a total of at least  $\frac{1}{2}$  below  $\mu^3$ .

It always allocates budget to intervals approved by a high number of voters, as long as the total size of these intervals is at most 1.

**Lemma 1.** *For any given instance, UTIL returns an aggregate  $a^U$  with the following property for any  $k \in [n]$ :*

$$\square(a^U, \mu^k) = \min \left( 1, \sum_{j \in [m]} \mu_j^k \right).$$

*Proof.* Let  $k \in [n]$ . It is easy to see that, when phantom  $f_{k-1}$  reaches 1, the median on each alternative  $j$  is exactly  $\mu_j^k$ .<sup>3</sup> Thus, if  $\sum_{j \in [m]} \mu_j^k < 1$ , then normalization has not been reached and  $a_j^U \geq \mu_j^k$  for all  $j \in [m]$ . On the other hand, if  $\sum_{j \in [m]} \mu_j^k \geq 1$ , then normalization is attained before or when  $f_{k-1}$  reaches 1 and thus  $a_j^U \leq \mu_j^k$  for all  $j \in [m]$ , implying  $\square(a^U, \mu^k) = \sum_{j \in [m]} a_j^U = 1$ .  $\square$

Using this lemma, we can now show Theorem 1.

*Proof of Theorem 1.* Let  $m, n \in \mathbb{N}$  and consider a profile  $P \in \mathcal{P}_{n,m}$  and a range-respecting mechanism  $\mathcal{A}$  satisfying proportional spending. We denote its aggregate by  $a = \mathcal{A}(P)$ . We prove the theorem by first showing a lower bound on the welfare achieved by such a mechanism and then giving an upper bound on the optimal welfare achieved by UTIL.

For aggregate  $a$ , we can compute the welfare as

$$w(a) = \sum_{i \in [n]} \sum_{j \in [m]} \min(a_j, p_{i,j}) = \sum_{k \in [n]} \sum_{j \in [m]} \min(a_j, \mu_j^k).$$

Let  $k^* \in [n]$  be the minimum number such that  $\sum_{j \in [m]} \mu_j^{k^*} \geq \frac{n-k^*+1}{n}$ . This is well-defined, since by the normalization of the votes we have  $\sum_{j \in [m]} \mu_j^n \geq 1 > \frac{1}{n}$ . By the proportional spending property, we know that for each  $k < k^*$ , it holds that  $a_j \geq \mu_j^k$  for all  $j \in [m]$ . For  $k \geq k^*$ , we get that  $\sum_{j \in [m]} \min(a_j, \mu_j^k) \geq \sum_{j \in [m]} \min(a_j, \mu_j^{k^*}) \geq \frac{n-k^*+1}{n}$ . We can thus bound the welfare of  $a$  in the following way:

<sup>3</sup>This expression is not defined for the last phantom ( $k = n + 1$ ), but normalization is always attained before that phantom moves, since  $\sum_{j \in [m]} \mu_j^n \geq 1$ .

$$\begin{aligned}
w(a) &= \sum_{k \in [n]} \sum_{j \in [m]} \min(a_j, \mu_j^k) \\
&= \sum_{k=1}^{k^*-1} \sum_{j \in [m]} \min(a_j, \mu_j^k) + \sum_{k=k^*}^{n-1} \sum_{j \in [m]} \min(a_j, \mu_j^k) + \sum_{j \in [m]} \min(a_j, \mu_j^n) \\
&\geq \sum_{k=1}^{k^*-1} \sum_{j \in [m]} \mu_j^k + \sum_{k=k^*}^{n-1} \frac{n-k^*+1}{n} + \sum_{j \in [m]} a_j = \left( \sum_{k=1}^{k^*-1} \sum_{j \in [m]} \mu_j^k \right) + (n-k^*) \frac{n-k^*+1}{n} + 1,
\end{aligned}$$

where the inequality follows from the arguments above and the range respect of  $\mathcal{A}$ .

By partitioning the welfare in a similar way for UTIL, we obtain

$$\begin{aligned}
w(a^U) &= \sum_{k \in [n]} \sum_{j \in [m]} \min(a_j^U, \mu_j^k) = \sum_{k=1}^{k^*-1} \sum_{j \in [m]} \min(a_j^U, \mu_j^k) + \sum_{k=k^*}^n \sum_{j \in [m]} \min(a_j^U, \mu_j^k) \\
&\leq \left( \sum_{k=1}^{k^*-1} \sum_{j \in [m]} \mu_j^k \right) + n - k^* + 1.
\end{aligned}$$

Here, the inequality follows by applying Lemma 1 for  $k < k^*$  and the normalization of  $a^U$ .

Combining the two bounds above, we can bound the welfare approximation as

$$\begin{aligned}
\frac{w(a^U)}{w(a)} &\leq \frac{\left( \sum_{k=1}^{k^*-1} \sum_{j \in [m]} \mu_j^k \right) + n - k^* + 1}{\left( \sum_{k=1}^{k^*-1} \sum_{j \in [m]} \mu_j^k \right) + (n-k^*) \frac{n-k^*+1}{n} + 1} \leq \frac{n - k^* + 1}{(n-k^*) \frac{n-k^*+1}{n} + 1} \\
&= \frac{n(n-k^*+1)}{(n-k^*)(n-k^*+1) + n} \leq \max_{\ell \in [n]} \frac{n\ell}{n + \ell(\ell-1)},
\end{aligned}$$

where the last inequality holds because  $k^* \in [n]$ , and thus  $n - k^* + 1 \in [n]$ . Since UTIL returns the welfare-maximizing allocation [Freeman et al., 2021] and the last expression is precisely  $\alpha^*(n)$ , this finishes the proof.  $\square$

We now show that UTILPROP satisfies proportional spending.

**Lemma 2.** *UTILPROP satisfies proportional spending.*

*Proof.* Let  $P \in \mathcal{P}_{n,m}$  be a profile. Let  $t$  be a time of normalization of UTILPROP and let  $k^* \in [n]_0$  be the highest index such that  $f_{k^*}(t) > 0$ . We observe for each  $1 \leq k \leq k^*$  that the median on each alternative  $j$  is at least  $\min\left(\frac{n-k+1}{n}, \mu_j^k\right)$ . This is because we have  $f_{k-1}(t) = \frac{n-k+1}{n}$  and therefore there are at most  $n-k+1$  phantoms and at most  $k-1$  voters strictly below  $\min\left(\frac{n-k+1}{n}, \mu_j^k\right)$ . Thus, for any  $k \leq k^*$  we have  $a_j \geq \min\left(\frac{n-k+1}{n}, \mu_j^k\right)$  for all  $j \in [m]$ , implying

$$\sqcap(a, \mu^k) \geq \sum_{j \in [m]} \min\left(\frac{n-k+1}{n}, \mu_j^k\right) \geq \min\left(\frac{n-k+1}{n}, \sum_{j \in [m]} \mu_j^k\right).$$

For the final inequality, note that if  $\mu_j^k > \frac{n-k+1}{n}$  for some  $j \in [m]$ , then  $\sum_{j \in [m]} \min\left(\frac{n-k+1}{n}, \mu_j^k\right) \geq \frac{n-k+1}{n}$ . Otherwise,  $\sum_{j \in [m]} \min\left(\frac{n-k+1}{n}, \mu_j^k\right) = \sum_{j \in [m]} \mu_j^k$ .

For  $k > k^*$ , we know that  $f_k(t) = 0$  and therefore the median on each alternative can be at most  $\mu_j^k$  (there are at most  $k$  phantoms and at most  $n - k$  voters strictly above  $\mu_j^k$ ). Thus,  $\square(a, \mu^k) = \sum_{j \in [m]} \min(a_j, \mu_j^k) = \sum_{j \in [m]} a_j = 1$  for each  $k > k^*$ .  $\square$

Freeman et al. [2021] showed that a moving-phantom mechanism which first moves the first phantom to position 1 while keeping all others at 0, and keeps the last phantom at 0 at all times, is range respecting. Since UTILPROP satisfies this property, UTILPROP is range respecting. We obtain the following corollary by combining this observation, Theorem 1 and Lemma 2.

**Corollary 1.** *UTILPROP attains a welfare approximation of  $\alpha^*(n)$ .*

Finally, we also show that GREEDYDECOMP satisfies proportional spending.

**Lemma 3.** *GREEDYDECOMP satisfies proportional spending.*

*Proof.* We first observe that, after the  $k^{\text{th}}$  (outer) iteration of GREEDYDECOMP, it must either be the case that  $N_j^+ = \emptyset$  or that  $N_j^+ \neq \emptyset$  and  $\tilde{a}_j = \mu_j^k$  for each alternative  $j$ , since the only reason GREEDYDECOMP does not increase  $\tilde{a}_j$  to  $\mu_j^k$  is if all voters voting above  $\tilde{a}_j$  have run out of budget. Also, in either case we have  $\tilde{a}_j \leq \mu_j^k$ .

Now suppose the lemma is false and let  $P \in \mathcal{P}_{n,m}$  be a profile for which there exists a  $k \in [n]$  with  $\square(a, \mu^k) < \min\left(\frac{n-k+1}{n}, \sum_{j \in [m]} \mu_j^k\right)$ . Then there must be some alternative  $j$  with  $a_j < \mu_j^k$ . Together with the above observation it follows that  $N_j^+$  is empty after the  $k^{\text{th}}$  iteration of GREEDYDECOMP. By definition, there are at least  $n - k + 1$  voters voting above or equal to  $\mu_j^k$  (and thus strictly above  $\tilde{a}_j \leq a_j$ ). If  $N_j^+$  is empty, then these  $n - k + 1$  voters must have zero budget, which implies that they have spent their total budget of  $\frac{n-k+1}{n}$  in previous iterations, i.e.,

$$\square(a, \mu^k) \geq \square(\tilde{a}, \mu^k) = \sum_{j \in [m]} \min(\tilde{a}_j, \mu_j^k) = \sum_{j \in [m]} \tilde{a}_j \geq \frac{n - k + 1}{n},$$

contradicting the initial assumption.  $\square$

Note that decomposability implies that a mechanism can never spend more than the highest vote on an alternative. Also, proportional spending implies that we can never spend less than the lowest vote. Thus, GREEDYDECOMP is also range respecting, and we conclude the following corollary from Theorem 1 and Lemma 3.

**Corollary 2.** *GREEDYDECOMP attains a welfare approximation of  $\alpha^*(n)$ .*

Together with the lower bound from Proposition 5, the matching upper bounds in Corollary 1 and Corollary 2 show that the price of single-minded proportionality is exactly  $\alpha^*(n)$ . Notably, this price does not increase even under the additional requirement of truthfulness (Corollary 1) and under the stronger fairness notion of decomposability (Corollary 2).

### 3.4 Welfare Approximation in Terms of $m$

Recall that Michorzewski et al. [2020] express their welfare approximations for the related problem of budget allocation under approval utilities as functions of  $m$ . In particular, they establish an upper bound on the price of fairness (for a different fairness notion) of  $\frac{m}{2\sqrt{m-2}}$ , as well as a lower bound of  $\frac{\sqrt{m}}{2}$ . As discussed when introducing our own lower bound, their lower bound also applies to the price of single-minded proportionality in our setting. Thus, orthogonally to our welfare bounds in terms of  $n$ , it is natural to ask whether a comparable bound in terms of  $m$  can be obtained here.

A key obstacle is that our techniques developed in this section provide guarantees for each level voter  $\mu^k$  via the notion of *proportional spending*. In contrast, the approach of Michorzewski et al. [2020, Theorem 2] builds upon guaranteeing welfare to supporters of individual alternatives. However, their approach does not transfer directly to our utility model, since voters in our model only “approve” an alternative up to a certain point. That said, a closer look at the proof of Lemma 2 reveals that UTILPROP in fact satisfies a stronger variant of proportional spending that holds *per alternative*. Using this observation and combining our arguments from the proof of Theorem 1 with the techniques of Michorzewski et al. [2020], we show that UTILPROP indeed achieves a welfare bound of  $\frac{m}{2\sqrt{m-2}}$ . We defer the proof to Appendix B.3. In the following statement, we say that a mechanism attains a welfare approximation of  $\beta(m)$ , if, for every  $n, m \in \mathbb{N}$  and every  $P \in \mathcal{P}_{n,m}$  it holds that  $\frac{w^{\text{OPT}}}{w(\mathcal{A}(P))} \leq \beta(m)$ .

**Theorem 2.** *UTILPROP attains a welfare approximation of  $\beta(m) = \frac{m}{2\sqrt{m-2}}$ .*

Unfortunately, GREEDYDECOMP does not satisfy our stronger notion of proportional spending, and we leave open the question of whether a similar bound can be achieved for GREEDYDECOMP.

## 4 Social Welfare of Truthful and Single-Minded Proportional Mechanisms

In this section, we define a welfare dominance relation among mechanisms and characterize the relationship of eight moving-phantom mechanisms. In particular, we show that UTILPROP is instance-wise welfare-optimal among all single-minded proportional moving-phantom mechanisms, and thereby among all known single-minded proportional and truthful mechanisms.

The definition of our relation  $\triangleright$  is simple: For two mechanisms  $\mathcal{A}$  and  $\mathcal{B}$ ,  $\mathcal{A} \triangleright \mathcal{B}$  indicates that, for every input profile,  $\mathcal{A}$  attains at least as much social welfare as  $\mathcal{B}$ . Moreover, we refer to a mechanism  $\mathcal{A}$  that has  $\mathcal{A} \triangleright \mathcal{B}$  (resp.  $\mathcal{B} \triangleright \mathcal{A}$ ) for every  $\mathcal{B}$  within a certain class of mechanisms as a *welfare-maximizing* (resp. *welfare-minimizing*) mechanism within this class. Although not all moving-phantom mechanisms are comparable under  $\triangleright$ , Lemma 4 enables us to order almost all moving-phantom mechanisms proposed in the literature, with only a single pair being incomparable. Theorem 3 states these comparisons; Figure 5 provides a visual aid.

Before stating the theorem, we define two additional moving-phantom mechanisms that we show to be welfare-minimizing within certain classes. First, CONSTANT is the trivial moving-phantom mechanism whose output is the uniform aggregate ( $\frac{1}{m}$  for every alternative) regardless of the profile. Second, FAN is the moving-phantom mechanism induced by the phantom system that moves all phantoms together from 0 to 1, but each phantom  $f_k$  stops moving once it reaches its final position  $\frac{n-k}{n}$ . Again, we refer to Appendix A for formal definitions and illustrations.

**Theorem 3.** *The following relations hold:*

- (i) *UTIL is welfare-maximizing and CONSTANT is welfare-minimizing among moving-phantom mechanisms.*
- (ii) *UTILPROP is welfare-maximizing and FAN is welfare-minimizing among single-minded proportional moving-phantom mechanisms.*
- (iii) *PIECEWISEUNIFORM  $\supseteq$  INDEPENDENTMARKETS and LADDER  $\supseteq$  INDEPENDENTMARKETS, but PIECEWISEUNIFORM and LADDER are incomparable according to  $\supseteq$ .*
- (iv) *FAN  $\supseteq$  GREEDYMAX.*

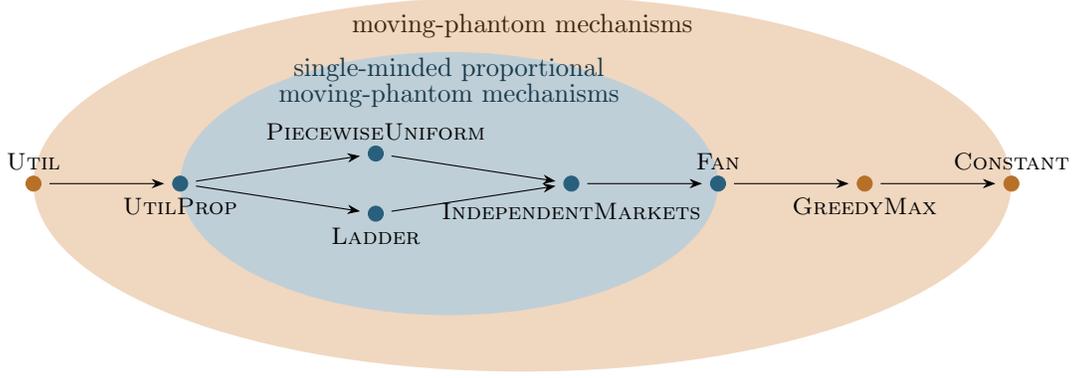


Figure 5: Welfare domination relationships among known moving-phantom mechanisms. The mechanisms UTILPROP, FAN, and the trivial CONSTANT mechanism are defined in this paper.

The rest of this section is devoted to proving Theorem 3. The main technical ingredient to compare the welfare that two moving-phantom mechanisms provide for a particular instance is given by the following structural lemma: If, at the time of normalization, the phantom positions of one system can be obtained from those of another by shifting all phantoms above some threshold upward and all phantoms below the threshold downward, then the former system yields weakly higher social welfare.

**Lemma 4.** *Let  $n \in \mathbb{N}$  be fixed,  $P = \{p_1, \dots, p_n\}$  be a profile and  $f_0, \dots, f_n \in [0, 1]$  and  $g_0, \dots, g_n \in [0, 1]$  the positions of the phantoms of two phantom systems  $\mathcal{F}_n$  and  $\mathcal{G}_n$  at a time of normalization (respectively) for  $P$ , i.e.,*

$$\sum_{j \in [m]} \text{med}(f_0, \dots, f_n, p_{1,j}, \dots, p_{n,j}) = \sum_{j \in [m]} \text{med}(g_0, \dots, g_n, p_{1,j}, \dots, p_{n,j}) = 1.$$

*Let  $a^{\mathcal{F}}$  and  $a^{\mathcal{G}}$  be the output allocations of the corresponding moving-phantom mechanisms. If there exists an index  $k^* \in \{-1, 0, \dots, n\}$  such that  $f_i \geq g_i$  for all  $i \in \{0, \dots, k^*\}$  and  $f_i \leq g_i$  for all  $i \in \{k^* + 1, \dots, n\}$ , then  $w(a^{\mathcal{F}}) \geq w(a^{\mathcal{G}})$ .*

While the proof of Lemma 4 is deferred to Appendix C.1, the key intuition is the following: The phantom placement of  $\mathcal{G}$  can be transformed into the phantom placement of  $\mathcal{F}$  by moving up those phantoms with index  $i \leq k^*$  and then moving down those phantoms with index  $i > k^*$ . The first movement increases the budget spent on intervals approved by at least  $n - k^*$  voters, the second movement decreases the budget spent on intervals approved by at most  $n - k^* - 1$  voters.

For the relations stated in Theorem 3, it suffices to show that we can apply Lemma 4 by showing that its condition is satisfied for any two points in time for the two corresponding phantom systems. We defer most of these proofs to Appendix C.2, where we also show the incomparability of PIECEWISEUNIFORM and LADDER by presenting two instances such that each mechanism strictly outperforms the other in terms of welfare in one of them.

The fact that UTILPROP is welfare-maximizing among single-minded proportional moving-phantom mechanisms, stated in item (ii), is particularly interesting and requires a deeper understanding of these mechanisms. To this end, we show that a condition that Freeman et al. [2021] showed to be sufficient for a moving-phantom mechanism to be single-minded proportional is also necessary: A moving-phantom mechanism is single-minded proportional if and only if it is induced by a phantom system such that each phantom  $k$  is at position  $\frac{n-k}{n}$  at time  $t = 1$ .

**Lemma 5.** *A moving-phantom mechanism  $\mathcal{A}$  is single-minded proportional if and only if there is a family of phantom systems,  $\mathcal{F}_n = \{f_k \mid k \in [n]_0\}$  for each  $n \in \mathbb{N}$ , that produces the same outcome as  $\mathcal{A}$  for every instance, with the property that for all  $n \in \mathbb{N}$  we have  $f_k(1) = \frac{n-k}{n}$  for each  $k \in [n]_0$ .*

The proof, deferred to Appendix C.3, works by modifying the phantom system of an arbitrary single-minded proportional moving-phantom mechanism, in such a way that no phantom  $k$  moves beyond position  $\frac{n-k}{n}$ . We show that this modification does not change the output for any instance. Using Lemmas 4 and 5, we can now prove that UTILPROP is welfare-maximizing among single-minded proportional moving-phantom mechanisms.

*Proof of Theorem 3 (ii) (first part).* Let  $n$  be fixed and  $\mathcal{F}_n$  be the phantom system of UTILPROP. By Lemma 5, for any single-minded proportional moving-phantom mechanism  $\mathcal{A}$ , we can find a phantom system  $\mathcal{G}_n$  inducing it such that for any phantom  $k$  of  $\mathcal{G}_n$ , we have  $g_k(1) = 1 - \frac{k}{n}$ . We show that for any two times  $t, t' \in [0, 1]$ , we can find a threshold index  $k^* \in \{-1, \dots, n\}$ , such that all phantoms with index  $k \leq k^*$  from  $\mathcal{F}_n$  lie above the ones of  $\mathcal{G}_n$  at times  $t$  and  $t'$ , respectively, and all the phantoms with index  $k > k^*$  lie below.

Let  $k'$  be the index of the lowest non-zero phantom of  $\mathcal{F}_n$  at time  $t$ . Note that, by the definition of UTILPROP, we have  $f_k(t) = 1 - \frac{k}{n} \geq g_k(t')$  for all  $k < k'$  and  $f_k(t) = 0 \leq g_k(t')$  for all  $k > k'$ . Now, we choose  $k^* = k'$  if  $f_{k'}(t) \geq g_{k'}(t')$  and  $k^* = k' - 1$  if  $f_{k'}(t) < g_{k'}(t')$ . In both cases,  $k^*$  satisfies the conditions from Lemma 4. Applying the lemma yields that UTILPROP always achieves at least the welfare that  $\mathcal{A}$  achieves, completing the proof.  $\square$

## 5 Social Welfare of Decomposable Mechanisms

In the previous sections, we analyzed the social welfare of single-minded proportional mechanisms and, in particular, showed that UTILPROP both guarantees the best-possible approximation ratio in the worst case and maximizes the welfare among single-minded proportional moving-phantom mechanisms for every instance. Here, we show that the situation is different for decomposable mechanisms: While GREEDYDECOMP gives the best-possible welfare approximation in the worst case, it does *not* maximize welfare among decomposable mechanisms on every instance and, in fact, no computationally efficient mechanism can do so, unless  $P=NP$ .

To make this statement formal, let UTILDECOMP be a mechanism that outputs, on every instance, some decomposable allocation maximizing social welfare, using an arbitrary but fixed tie-breaking rule. To gain intuition, we begin with a simple example showing that GREEDYDECOMP may return

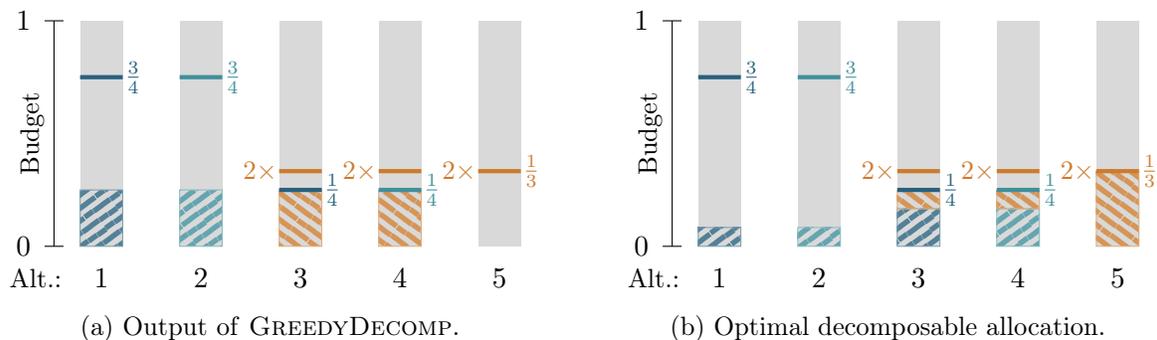


Figure 6: Visualization of the profile and aggregates from Example 1 (Blue) upwards hatched areas represent contributions of voters 1 and 2, (orange) downwards hatched areas are contributions of voters 3 and 4.

an allocation with strictly lower welfare than UTILDECOMP.<sup>4</sup>

**Example 1.** Consider the instance with  $n = 4$  voters and  $m = 5$  alternatives and profile

$$P = \begin{pmatrix} 3/4 & 0 & 1/4 & 0 & 0 \\ 0 & 3/4 & 0 & 1/4 & 0 \\ 0 & 0 & 1/3 & 1/3 & 1/3 \\ 0 & 0 & 1/3 & 1/3 & 1/3 \end{pmatrix},$$

which we illustrate in Figure 6.

GREEDYDECOMP first distributes the budget from voters 3 and 4 over the alternatives 3 and 4 and then has to spend the budget of voters 1 and 2 on alternatives 1 and 2 (compare Figure 6a), leading to the allocation  $a = (\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, 0)$  with welfare  $w(a) = 2 \cdot \frac{1}{4} + 2 \cdot 3 \cdot \frac{1}{4} = 2$ .

UTILDECOMP outputs  $a' = (\frac{1}{12}, \frac{1}{12}, \frac{1}{4}, \frac{1}{4}, \frac{1}{3})$  with welfare  $w(a') = 2 \cdot \frac{1}{12} + 2 \cdot 3 \cdot \frac{1}{4} + 2 \cdot \frac{1}{3} = \frac{7}{3} > 2$ . To see, that  $a'$  is decomposable, consider the voter contributions (compare Figure 6b)

$$c = \begin{pmatrix} 1/12 & 0 & 1/6 & 0 & 0 \\ 0 & 1/12 & 0 & 1/6 & 0 \\ 0 & 0 & 1/12 & 0 & 1/6 \\ 0 & 0 & 0 & 1/12 & 1/6 \end{pmatrix}.$$

In the following, we prove that computing a welfare-optimal decomposable allocation (and hence the outcome of UTILDECOMP) is not possible in polynomial time unless  $P = NP$ . Nevertheless, a welfare-optimal decomposable allocation can be found via integer linear programming; see Appendix D.2. To formalize our hardness result, we introduce the following decision problem.

**Definition 1** (DECOMPOSABLEWELFARETHRESHOLD (DWT)). An instance consists of a profile  $P$  and a rational number  $C \in \mathbb{Q}$ . The decision problem asks whether there exists a decomposable allocation  $a$  and voter contributions  $(c_{i,j})_{i \in [n], j \in [m]}$  such that  $a$  has social welfare  $w(a) \geq C$ .

Clearly, DWT is contained in NP. We show that DWT is NP-complete, which implies that computing the outcome of UTILDECOMP is NP-hard. While the proof of this result is deferred to Appendix D.3, we sketch here the main ideas involved.

<sup>4</sup>We further show in Appendix D.1 that there are instances on which GREEDYDECOMP and UTILDECOMP output the same allocation, and that allocation is Pareto-dominated by a non-decomposable one.

**Theorem 4.** *DECOMPOSABLEWELFARETHRESHOLD is NP-complete. In particular, computing the outcome of UTILDECOMP is NP-hard.*

*Proof Sketch.* The proof reduces from EXACTCOVERBY3SETS (X3C), which asks whether a set of elements  $U$  of size  $3q$  admits a set cover of size  $q$  from among a family of 3-element subsets  $\mathcal{S} = \{S_1, \dots, S_k\}$  of  $U$ . It includes one *subset alternative* for each  $S_j$  and  $3q$  *element voters* whose preferences encode the incidence structure: for each element  $e \in U$ , the corresponding voter assigns a small positive vote<sup>5</sup>  $\varepsilon$  to those  $S_j$  that contain  $e$  and zero to all other subset alternatives. A collection of auxiliary *toggle voters* each vote highly for a single subset alternative as well as for a commonly liked alternative  $Y$ . Due to decomposability constraints,  $Y$  can absorb the spending of exactly  $q$  toggle voters. This allows exactly  $q$  subset alternatives to be “turned on”, while the remaining  $k - q$  subset alternatives remain “turned off” because toggle voter spending crowds out potential element voter contributions. Finally, additional auxiliary voters ensure that each element voter has  $\varepsilon/3$  of their budget that yields higher welfare when spent on a subset alternative than on any other alternative permitted by decomposability.

Hardness follows from showing that, subject to decomposability, social welfare is strictly higher on X3C instances that admit an exact cover than those that do not. If an exact cover exists, then it is possible to turn on  $q$  subset alternatives such that each receives  $\varepsilon/3$  contribution from exactly three element voters, yielding high welfare. If no exact cover exists, then for any  $q$  turned-on subset alternatives there must exist an element voter who assigns zero weight to all of them; decomposability then forces this voter to contribute their entire budget to an alternative with lower marginal contribution to welfare, thus reducing total welfare relative to the exact-cover case.  $\square$

Given the hardness of computing the optimal decomposable allocation for a given instance, approximating it is, in a sense, the best we can hope for. We finish with a positive result for GREEDYDECOMP in this regard: For every instance, it returns an allocation whose welfare is at least half of the welfare of any other decomposable allocation. The proof is deferred to Appendix D.4, but we give some intuition in a proof sketch.

**Theorem 5.** *For any profile with  $n \geq 2$  voters, UTILDECOMP achieves at most  $2 - \frac{1}{n-1}$  times the welfare that GREEDYDECOMP achieves.*

*Proof Sketch.* Given a profile  $P$ , we compare the allocation  $a^{\text{GD}}$  of GREEDYDECOMP to a welfare-optimal decomposable allocation  $a^{\text{UD}}$ . We consider voter contributions certifying the decomposability of  $a^{\text{UD}}$  and imagine that voters spend their contributions in some arbitrary fixed order. This way, every marginal amount that  $a^{\text{UD}}$  spends on some  $j \in [m]$  can be assigned to a specific voter.

Our goal is to show that the welfare advantage of  $a^{\text{UD}}$  over  $a^{\text{GD}}$  on alternatives where  $a_j^{\text{UD}} > a_j^{\text{GD}}$  is bounded by the welfare that  $a^{\text{GD}}$  achieves. To this end we define  $a^{\text{UD} \cap \text{GD}} = (\min(a_j^{\text{UD}}, a_j^{\text{GD}}))_{j \in [m]}$  as the overlapping budget spending and consider an alternative  $j$  on which  $a_j^{\text{UD}} > a_j^{\text{GD}}$ . Let  $i$  be a voter who contributes to buying more of alternative  $j$  than GREEDYDECOMP does. By the definition of decomposability, that voter’s vote exceeds  $a_j^{\text{GD}}$  on  $j$ . Thus, during the execution of GREEDYDECOMP, voter  $i$  has already run out of budget when the algorithm considers spending more than  $a_j^{\text{GD}}$  on alternative  $j$ . In turn, this means that voter  $i$  must have spent their entire budget where it is approved by at least as many voters as approve the marginal increase from  $a_j^{\text{GD}}$  to  $a_j^{\text{UD}}$ . By generalizing this argument over all voters and alternatives  $j$  with  $a_j^{\text{UD}} > a_j^{\text{GD}}$ , we can

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<sup>5</sup>For technical reasons that are consistent with the intuition provided here, the full proof actually uses  $\varepsilon + \frac{1}{n}$ , not  $\varepsilon$ .

show that indeed the welfare achieved by  $a^{\text{UD}}$  over  $a^{\text{GD}}$  is bounded by the welfare of  $a^{\text{GD}}$ , i.e.,  $w(a^{\text{UD}}) - w(a^{\text{UD} \cap \text{GD}}) \leq w(a^{\text{GD}})$ . Then,

$$\frac{w(a^{\text{UD}})}{w(a^{\text{GD}})} = \frac{w(a^{\text{UD}}) - w(a^{\text{UD} \cap \text{GD}}) + w(a^{\text{UD} \cap \text{GD}})}{w(a^{\text{GD}})} \leq \frac{w(a^{\text{GD}}) + w(a^{\text{GD}})}{w(a^{\text{GD}})} = 2.$$

The additional term  $-\frac{1}{n-1}$  in the approximation factor is achieved by observing that, by the range-respect property, GREEDYDECOMP never spends budget where no voter approves it. This gives a lower bound on the welfare that GREEDYDECOMP achieves over  $a^{\text{UD}}$  on alternatives where it spends more, which can be used in the inequality above to slightly improve the bound.  $\square$

In Appendix D.5, we generalize the construction from Example 1 and give instances on which UTILDECOMP has  $2 - \Theta(1/n)$  times the welfare that GREEDYDECOMP achieves, showing that the analysis above is asymptotically tight. Since the profile used in the proof has  $m = n + 1$ , the instance also yields a lower bound of  $2 - \Theta(1/m)$  for the welfare approximation. Beyond this specific mechanism, the question of the best approximation of the welfare provided by UTILDECOMP that can be guaranteed by a decomposable, polynomial-time mechanism remains open.

## 6 Discussion

We have studied budget aggregation under  $\ell_1$  utilities from the dual perspectives of proportionality and social welfare. By deriving a tight bound on the price of single-minded proportionality, our results quantify the welfare cost of enforcing fairness in this setting. Moreover, this cost (in terms of  $n$ ) does not increase even when single-minded proportionality is combined with truthfulness or strengthened to decomposability. Beyond worst-case guarantees, by introducing an instance-wise welfare dominance relation, we identify UTILPROP as welfare-optimal among single-minded proportional moving-phantom mechanisms. Finally, we have shown that while the welfare-optimal decomposable outcome is NP-hard to compute, a simple greedy mechanism GREEDYDECOMP achieves a 2-approximation. We discuss extensions and open questions below.

**Weighted Setting** A natural generalization of the budget aggregation setting is *donor coordination* [Brandl et al., 2021, Brandt et al., 2023], where voters contribute different amounts to a pool of money that is to be distributed among several charities. We can extend our model to capture such entitlements by assigning each voter  $i$  an integer weight  $\omega_i$ . In Appendix E we discuss how to generalize the mechanisms and axioms studied in this paper to the weighted setting and demonstrate that all results of this paper extend. In particular, we show that the class of weighted moving-phantom mechanisms is truthful, that all our welfare-approximation bounds still hold when replacing the parameter  $n$  by the total weight of all voters  $b$ , and that all instance-wise welfare domination results remain true for the weighted versions of the moving-phantom mechanisms.

**Welfare Approximation for Other Proportional Mechanisms** In Section 3, we showed that UTILPROP attains the optimal welfare approximation among single-minded proportional mechanisms, namely  $\Theta(\sqrt{n})$ . This guarantee does not hold for PIECEWISEUNIFORM, INDEPENDENTMARKETS, or FAN. Indeed, in Appendix B.4 we establish a lower bound, linear in  $n$ , on the welfare approximation of PIECEWISEUNIFORM; the same bound applies to INDEPENDENTMARKETS and FAN by Theorem 3. For LADDER, we do not have a definitive answer. However, we can show that it does not satisfy proportional spending (see Appendix B.4). Hence, a fundamentally different proof

technique would be required to establish worst-case optimality for this mechanism, if it holds at all. Regardless, the instance-wise welfare domination of UTILPROP over LADDER establishes the optimality of UTILPROP in a stronger sense among single-minded proportional mechanisms.

**Phantom Lattice** Consider the welfare-dominance relation defined in Section 4. Interestingly, for both the class of moving-phantom mechanisms and the class of single-minded moving-phantom mechanisms, we can identify a welfare-maximizing and a welfare-minimizing moving-phantom mechanism. Since our dominance relation is, by construction, a partial order, the existence of such extremal elements was not a priori guaranteed and may therefore be viewed as surprising. Their existence naturally leads to the following open question: Does the dominance relation form a *lattice*, either within the set of all moving-phantom mechanisms or within the set of single-minded proportional moving-phantom mechanisms? While we do not settle this question in full generality, we show in Appendix C.4 that both statements hold in the special case of two alternatives.

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## A Overview of the Moving-Phantom Mechanisms from the Literature

Table 1: Overview over the moving-phantom mechanisms from the literature.

Name Definition	Visualization	Description
<p><b>CONSTANT</b></p> $f_k(t) = t$		<p>All phantoms move simultaneously. Normalization is always reached when all phantoms reach position <math>\frac{1}{m}</math>.</p>
<p><b>GREEDYMAX</b> [de Berg et al., 2024]</p> $f_k(t) = t \cdot \min(1, n - k)$		<p>The top <math>n</math> phantoms move simultaneously, while the last one stays at zero.</p>
<p><b>FAN</b></p> $f_k(t) = \min\left(\frac{n - k}{n}, t\right)$		<p>All phantoms move simultaneously with each phantom stopping at position <math>\frac{n-k}{n}</math>.</p>
<p><b>INDEPENDENTMARKETS</b> [Freeman et al., 2021]</p> $f_k(t) = t \cdot \frac{n - k}{n}$		<p>All phantoms move simultaneously but at different speeds, until each phantom <math>k</math> reaches its final position <math>\frac{n-k}{n}</math> at time <math>t = 1</math>.</p>

Table 1: Overview over the moving-phantom mechanisms from the literature.

Name Definition	Visualization	Description
<p><b>LADDER</b> [Freeman and Schmidt-Kraepelin, 2024]</p> $f_k(t) = \max\left(t - \frac{k}{n}, 0\right)$		<p>All phantoms move at the same speed, but start at different times, such that each phantom <math>k</math> reaches its final position <math>\frac{n-k}{n}</math> at time <math>t = 1</math>.</p>
<p><b>PIECEWISEUNIFORM</b> [Caragiannis et al., 2024]</p> <p>For <math>t &lt; \frac{1}{2}</math></p> $f_k(t) = \begin{cases} \frac{4t(n-k)}{n} - 2t & \text{if } \frac{k}{n} \leq \frac{1}{2}, \\ 0 & \text{if } \frac{k}{n} > \frac{1}{2}, \end{cases}$ <p>and for <math>t \geq \frac{1}{2}</math></p> $f_k(t) = \begin{cases} \frac{(n-k)(3-2t)}{n} - 2 + 2t & \text{if } \frac{k}{n} \leq \frac{1}{2}, \\ \frac{(n-k)(2t-1)}{n} & \text{if } \frac{k}{n} > \frac{1}{2}. \end{cases}$		<p>In the first phase the uppermost half of the phantoms are spread uniformly across <math>[0, 1]</math>. In the second phase the lower half of the phantoms are spread across <math>[0, \frac{1}{2}]</math> while concentrating the upper phantoms into <math>[\frac{1}{2}, 1]</math>.</p>
<p><b>UTIL</b> [Freeman et al., 2021]</p> $f_k(t) = \max(0, \min(1, t(n+1) - k))$		<p>The phantoms move consecutively from zero to one.</p>
<p><b>UTILPROP</b></p> $f_k(t) = \max\left(0, \min\left(\frac{n-k}{n}, t(n+1) - k\right)\right)$		<p>The phantoms move consecutively from zero to <math>\frac{n-k}{n}</math>.</p>

## B Omitted Material from Section 3

### B.1 Proof of Proposition 3

**Proposition 3.** *GREEDYDECOMP is a decomposable budget aggregation mechanism and can be computed in polynomial time.*

*Proof.* We first show that after termination of GREEDYDECOMP, each voter  $i$  has a remaining budget of  $b_i = 0$ . Since each voter starts with a budget of  $\frac{1}{n}$  and every amount that is subtracted from the voters budget is added to the allocation, this then implies that GREEDYDECOMP returns a valid allocation after it terminates. Suppose there is some voter  $i$  with  $b_i > 0$  and therefore  $\sum_{j \in [m]} a_j < 1$  after termination. Let  $j$  be an alternative with  $p_{i,j} > a_j$  (such an alternative must exist, because votes are normalized). Choose  $k$  such that  $\mu_j^k = p_{i,j}$ . Then in the  $k^{\text{th}}$  outer iteration and after the last inner iteration with  $\tau^* = 1$ , we have  $p_{i,j} > a_j \geq \tilde{a}_j$ , which means  $N_j^+$  is non-empty.

But then the payment for some voter  $i' \in N_j^+$  and alternative  $j$  is  $\pi_{i',j} = \max\left(0, \frac{\min(\mu_j^k, \tau^*) - \tilde{a}_j}{|N_j^+|}\right) \geq \frac{\min(\mu_j^k, \tau^*) - \tilde{a}_j}{|N_j^+|} = \frac{p_{i',j} - \tilde{a}_j}{|N_j^+|} > 0$  and thus  $\tilde{a}_j$  would have been increased to some value strictly above  $a_j$ .

We now show that GREEDYDECOMP is decomposable. We can decompose the output of GREEDYDECOMP by tracking the amount of budget each voter pays for each alternative during the computation of GREEDYDECOMP. More formally, we can initialize  $c_{i,j} = 0$  for all voters  $i \in [n]$  and alternatives  $j \in [m]$  and insert  $c_{i,j} \leftarrow c_{i,j} + \pi_{i,j}(\tau^*)$  after Line 9. Since the budget of each voter  $i$  is fully exhausted (as shown above), we know that  $\sum_{j \in [m]} c_{i,j} = \frac{1}{n}$  and it only remains to show that no voter  $i$  is charged for an alternative  $j$  with  $a_j > p_{i,j}$ . Let  $i \in [n]$  be a voter with  $c_{i,j} > 0$  for some alternative  $j$ . Consider an inner iteration in which  $c_{i,j}$  is increased and let  $k$  be the index of the current outer iteration. In that iteration voter  $i$  has a positive payment  $\pi_{i,j}(\tau^*) > 0$ , which implies  $i \in N_j^+$ . When  $i \in N_j^+$  in outer iteration  $k$ , we know that  $p_{i,j} > \tilde{a}_j$  and  $\tilde{a}_j \geq \mu_j^{k-1}$  (by the invariant of the algorithm explained in Section 3.2). By the definition of  $\mu$ , there is no vote between  $\mu_j^{k-1}$  and  $\mu_j^k$  and therefore  $p_{i,j} \geq \mu_j^k$ . By the definition of  $N_j^+$ , there can be no voter with positive budget voting above  $p_{i,j}$  on alternative  $j$ , which means that  $\tilde{a}_j$  can never be increased above  $p_{i,j}$ .

Finally, we show that GREEDYDECOMP always terminates and that the output can be computed in polynomial time. In every inner iteration of the algorithm, at least (i) one voter's budget is exhausted ( $b_i = \sum_{j \in [m]} \pi_{i,j}(\tau^*) > 0$  for some voter  $i$ ), or (ii)  $\tau^* = 1$  and the next outer iteration starts. Also note that each voter can only exhaust their budget in a single iteration. Thus, we can have no more than  $2n$  inner iterations in total. To show that GREEDYDECOMP is computable in polynomial time, it only remains to show that, in each iteration, we can compute the next value of  $\tau^*$  efficiently. To do so, we iteratively try out different values  $\tau$ , until a voter overpays or  $\tau = 1$  is reached. Say we are in the  $k^{\text{th}}$  outer iteration with a current allocation  $\tilde{a}$ , sets  $N_j^+$  for each  $j \in [m]$ , and voter budgets  $b_i$  for each  $i \in [n]$ . We compute the payments for all voters for the values  $\tau \in \{0, 1\} \cup \{\tilde{a}_j \mid j \in [m]\} \cup \{\mu_j^k \mid j \in [m]\}$ . If we have  $b_i \geq \sum_{j \in [m]} \pi_{i,j}(1)$  for all voters  $i \in [n]$ , then we set  $\tau^* = 1$ , as no voter overpays. Otherwise, we identify the two lowest consecutive values  $\tau_1, \tau_2$  from the set with  $\sum_{j \in [m]} \pi_{i,j}(\tau_1) < b_i \leq \sum_{j \in [m]} \pi_{i,j}(\tau_2)$  for some voter  $i$  with  $b_i > 0$ . Note that, for all  $j \in [m]$  and all such voters  $i$ , the payment  $\pi_{i,j}(\tau) = \max\left(0, \frac{\min(\mu_j^k, \tau^*) - \tilde{a}_j}{|N_j^+|}\right)$  is linear within the interval  $[\tau_1, \tau_2]$ , as we do not cross  $\tilde{a}_j$  or  $\mu_j^k$ . Therefore, we can compute the value  $\tau_i^*$  for which each

such voter  $i$  runs out of budget as

$$\tau_i^* = \tau_1 + \frac{b_i - \sum_{j \in [m]} \pi_{i,j}(\tau_1)}{\sum_{j \in [m]} \pi_{i,j}(\tau_2) - \sum_{j \in [m]} \pi_{i,j}(\tau_1)} (\tau_2 - \tau_1).$$

Setting  $\tau^* = \min_{i \in [n]} \tau_i^*$  yields the minimum value for which some voter runs out of budget.  $\square$

We observe that, in addition to decomposability, GREEDYDECOMP satisfies neutrality and anonymity because it treats voters and alternatives symmetrically when iteratively allocating the budget.

## B.2 Proof of Proposition 4

**Proposition 4.** *For every  $n \in \mathbb{N}$ , we have  $\alpha^*(n) \leq \frac{n}{2\sqrt{n}-1}$ .*

*Proof.* We fix  $n \in \mathbb{N}$  and define the function  $h: [0, n] \rightarrow \mathbb{R}_+$  by

$$h(\ell) = \frac{n\ell}{n + \ell(\ell - 1)} \quad \text{for every } \ell \in [0, n].$$

Its derivative is given by

$$h'(\ell) = \frac{n(n + \ell(\ell - 1)) - n\ell(2\ell - 1)}{(n + \ell(\ell - 1))^2} = \frac{n(n - \ell^2)}{(n + \ell(\ell - 1))^2},$$

so  $h$  is increasing for  $\ell \in [0, \sqrt{n})$ , decreasing for  $\ell \in (\sqrt{n}, n]$ , and attains its maximum at  $\ell^* = \sqrt{n}$ , with

$$h(\sqrt{n}) = \frac{n\sqrt{n}}{n + \sqrt{n}(\sqrt{n} - 1)} = \frac{n}{2\sqrt{n} - 1}. \quad \square$$

## B.3 Proof of Theorem 2

**Theorem 2.** *UTILPROP attains a welfare approximation of  $\beta(m) = \frac{m}{2\sqrt{m}-2}$ .*

*Proof.* We refer to Figure 7 for a running illustration of this proof. However, note that for illustration purposes, votes and aggregate are overnormalized in this example, which is not possible for a valid profile. Let  $\{f_k\}_{k \in [n]_0}$  be the phantom system of UTILPROP and  $\{g_k\}_{k \in [n]_0}$  be the phantom system of UTIL. Fix a preference profile  $P \in \mathcal{P}_{n,m}$  and let  $t$  be the smallest point of normalization of UTILPROP. Let  $a$  be the outcome of UTILPROP and  $a^U$  be the outcome of UTIL. Now, let  $\hat{k} \in [n]_0$  be the maximum index such that  $f_{\hat{k}}(t) > 0$ . Intuitively, this is the last phantom that started moving before normalization is reached (compare Figure 7). Moreover, let  $k^*$  be the smallest  $k^* \in [\hat{k} - 1]$  such that  $\mu_j^{k^*+1} \geq \frac{n-k^*}{n}$  for any  $j \in [m]$  (in Figure 7,  $k^* = 2$ , because of the third lowest voter on the leftmost alternative). The interpretation of  $k^*$  is that it is the first phase of the phantom system for which UTIL and UTILPROP (may) behave differently. This is because if phantom  $g_{k^*}$  moves to 1 in UTIL, this corresponds to  $\mu^{k^*+1}$  being selected at this point in time. Now, recall that UTILPROP moves phantoms  $f_k$  to  $\frac{n-k}{n}$ . Hence, for every  $k < k^*$ , since  $\mu_j^{k+1} \leq \frac{n-k}{n} \leq 1$  for all  $j \in [m]$ , UTILPROP spends at least  $\mu_j^{k+1}$ , just as UTIL does. We remark that, if such an index does not exist, then  $a = a^U$  and there is nothing to be shown. Therefore, our first observation is that

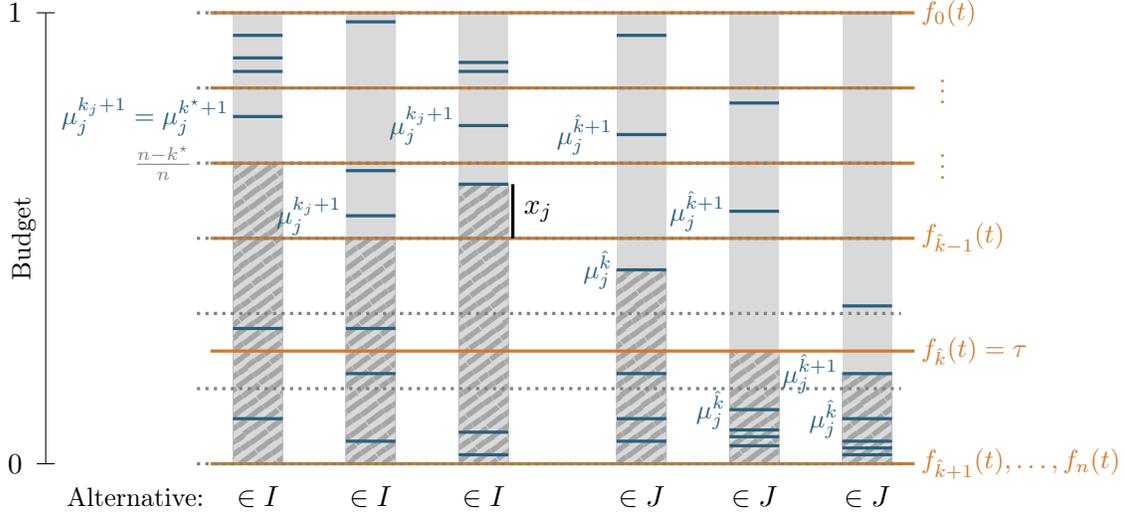


Figure 7: Illustration of the situation in the proof of Theorem 2. For this example, we have  $n = 6$  voters. Phantom  $f_{\hat{k}} = f_4$  is the last one that moved and the index  $k^*$  is 2, since on the leftmost alternative the third lowest voter votes above  $\frac{n-2}{n} = \frac{4}{6}$ . For illustration purposes, the votes and aggregate are not normalized in this example, which has the side effect that  $q = \sum_{j \in I} n_j > n$ , which is not possible for a valid profile.

the contribution to social welfare derived from votes in the levels  $\mu^1, \dots, \mu^{k^*}$  is, for both UTIL and UTILPROP, exactly

$$\sum_{k=1}^{k^*} \sum_{j \in [m]} \mu_j^k, \quad (1)$$

which holds because when phantom  $k^* - 1$  reaches its final position it holds that the median on each coordinate  $j$  is at least  $\mu_j^{k^*}$ , and therefore  $a_j \geq \mu_j^{k^*}$ . We will make use of this bound later. Note that the above expression only makes use of utility derived from votes in the levels  $\mu^1, \dots, \mu^{k^*}$ . To avoid double counting, in the following we are going to make use of utility generated from votes in the levels  $\mu^{k^*+1}, \dots, \mu^n$ .

Now, we divide the alternatives into two subgroups. First,  $I \subseteq [m]$  includes all alternatives  $j$  such that  $\mu_j^{k+1} \geq \frac{n-k}{n}$  for some  $k \in \{k^*, \dots, \hat{k} - 1\}$ ; we define  $k_j$  as the smallest such index for each  $j \in I$ . We also define  $\ell = |I|$  and  $J = [m] \setminus I$  for the other group.

### Bound for alternatives in $I$

We claim that  $\mu_j^{k_j+1} \geq a_j \geq \frac{n-k_j}{n}$  for any  $j \in I$ . This holds because, at time  $t$ , (i)  $\mu_j^{k_j+1}$  has at most  $n - k_j - 1$  voters strictly above and at most  $k_j$  phantoms strictly above, and (ii)  $\frac{n-k_j}{n}$  has  $k_j + 1$  phantoms weakly above and at least  $n - k_j$  voters weakly above. We define  $n_j = n - k_j$  and  $x_j \geq 0$  such that  $a_j = \frac{n_j}{n} + x_j$  (compare third alternative in Figure 7). Now, note that there are  $n_j$  voters within the levels  $\mu_j^{k_j+1}, \dots, \mu_j^n$  and all of these voters vote weakly above  $a_j$ . Hence, the contribution to their utility from alternatives in  $I$  is at least

$$\sum_{j \in I} n_j a_j = \sum_{j \in I} \frac{(n_j)^2}{n} + \sum_{j \in I} n_j x_j. \quad (2)$$

Before we continue, we define  $q = \sum_{j \in I} n_j$  and claim that  $q < n$ . To see why, note that  $\sum_{j \in I} \frac{n_j}{n} = \frac{q}{n}$  corresponds to the sum of the medians on alternatives in  $I$  at the time when phantom  $\hat{k} - 1$  reaches its final position. At this point, normalization is not yet reached and hence  $1 > \frac{q}{n}$ .

### Bound for alternatives in $J$

For the group  $J$ , we remark that  $a_j \in [\mu_j^{\hat{k}}, \mu_j^{\hat{k}+1}]$ , which holds because none of the final phantom positions of the phantoms in  $\{0, 1, \dots, \hat{k} - 1\}$  were restrictive for these alternatives, and on all alternatives in  $J$  we could move up all phantoms in  $\{0, 1, \dots, \hat{k} - 1\}$  to 1 without changing the outcome (illustrated on the last three alternatives in Figure 7).

For every  $j \in J$ , we define  $N_j$  as the set of voters satisfying the following two criteria: (i) the voter appears in  $\mu_j^{\hat{k}}, \dots, \mu_j^{\hat{k}+1}$  and (ii) votes weakly above  $a_j$ . For any  $j \in J$ , let  $\hat{n}_j = |N_j|$ . Moreover, define  $\tau = f_{\hat{k}}(t)$ , i.e., the position of the last moving phantom at the point of normalization. Note that  $a_j = \min(\max(\tau, \mu_j^{\hat{k}}), \mu_j^{\hat{k}+1})$  and

$$\hat{n}_j = \begin{cases} n - \hat{k} + 1 & \text{if } \tau \leq \mu_j^{\hat{k}} \\ n - \hat{k} & \text{if } \tau > \mu_j^{\hat{k}}. \end{cases}$$

As a direct consequence, all alternatives  $j \in J$  with *high*  $\hat{n}_j$  (i.e.,  $\hat{n}_j = n - \hat{k} + 1$ ) have the property that  $a_j \geq \tau$ , while all the alternatives with *low*  $\hat{n}_j$  (i.e.,  $\hat{n}_j = n - \hat{k}$ ) have the property that  $a_j \leq \tau$ . Hence, we can relabel the alternatives in  $J$  such that  $\hat{n}_1 \geq \dots \geq \hat{n}_{m-\ell}$  and  $a_1 \geq \dots \geq a_{m-\ell}$ . Now, applying Chebyshev's sum inequality, we obtain the following lower bound for the utility contribution of agents in  $N_j$  for any  $j \in J$ :

$$\sum_{j \in J} \hat{n}_j a_j \geq \frac{\sum_{j \in J} \hat{n}_j}{m - \ell} \sum_{j \in J} a_j. \quad (3)$$

We now claim that  $\sum_{j \in J} \hat{n}_j \geq n - q$ . To see why, note that all agents that were not counted in any of the  $n_j$  for  $j \in I$  vote weakly below  $a_j$  for all  $j \in I$ . Hence, such a voter votes at least  $1 - \sum_{j \in I} a_j$  on the alternatives in  $J$ . However, we also know that  $\sum_{j \in I} a_j + \sum_{j \in J} \mu_j^{\hat{k}} < 1$ , which holds because this is the state of the medians after phantom  $\hat{k} - 1$  moved to its final position, but at this point, normalization is not yet reached. Hence, each of the  $n - q$  voters must appear within the levels  $\mu_j^{\hat{k}+1}, \dots, \mu_j^{\hat{k}}$  for some  $j \in J$ , which establishes the claim.

Hence, we get the following lower bound for the contribution of agents in  $N_j$ :

$$\sum_{j \in J} \hat{n}_j a_j \geq \frac{n - q}{m - \ell} \sum_{j \in J} a_j. \quad (4)$$

We define  $n_j = \frac{n - q}{m - \ell}$  for all  $j \in J$  and observe that for any  $j' \in I, j \in J$  it holds that  $n_{j'} \geq n - \hat{k} + 1 \geq \frac{\sum_{j'' \in J} \hat{n}_{j''}}{m - \ell} \geq \frac{n - q}{m - \ell} = n_j$ . Hence, note that  $n^* := n - k^* = \max_{j \in J} (n_j) = \max_{j \in [m]} (n_j)$ .

Putting the above together with (2) and (4), we can lower-bound the utility of agents in levels

$\mu^{k^*}, \dots, \mu^n$  across all alternatives by:

$$\begin{aligned}
\sum_{j \in I} \frac{(n_j)^2}{n} + \frac{n-q}{m-\ell} \left( \sum_{j \in J} a_j + \sum_{j \in I} x_j \right) &= \sum_{j \in I} \frac{(n_j)^2}{n} + \frac{n-q}{m-\ell} \left( \frac{n-q}{n} \right) \\
&= \sum_{j \in I} \frac{(n_j)^2}{n} + \frac{n-q}{m-\ell} \left( \frac{(m-\ell)(n-q)}{m-\ell} \cdot \frac{1}{n} \right) \\
&= \sum_{j \in I} \frac{(n_j)^2}{n} + \sum_{j \in J} \frac{(n_j)^2}{n}, \tag{5}
\end{aligned}$$

where the last equality follows from the fact that  $|J| = m - \ell$ . We derive a lower bound for (5) in dependence on  $n^*$ . To this end, the next paragraph will closely follow the arguments by Michorzewski et al. [2020]. We define  $j^* \in \arg \max_{j \in [m]} (n_j)$  and  $s = \sum_{j \in [m] \setminus j^*} (\frac{n_j}{n^*})$ . By the Cauchy-Schwarz inequality we obtain that  $\sum_{j \in [m] \setminus j^*} (\frac{n_j}{n^*})^2 \geq \frac{s^2}{m-1}$ . Now, we can derive the following bound:

$$\sum_{j \in [m]} \frac{(n_j)^2}{n} = \frac{\sum_{j \in [m]} (n_j)^2}{\sum_{j \in [m]} n_j} = n^* \cdot \frac{\sum_{j \in [m]} (\frac{n_j}{n^*})^2}{\sum_{j \in [m]} \frac{n_j}{n^*}} = n^* \cdot \frac{1 + \sum_{j \in [m] \setminus j^*} (\frac{n_j}{n^*})^2}{1 + \sum_{j \in [m] \setminus j^*} (\frac{n_j}{n^*})} \geq n^* \cdot \frac{m-1+s^2}{(m-1)(1+s)}$$

Michorzewski et al. [2020] argue that this function (excluding the factor  $n^*$ ) for  $s \in [0, m-1]$  can be lower bounded by  $\frac{2\sqrt{m-2}}{m}$ . We can now start to put everything together:

$$\frac{w(a^U)}{w(a)} \leq \frac{\sum_{k=1}^{k^*} \sum_{j \in [m]} \mu_j^k + n - k^*}{\sum_{k=1}^{k^*} \sum_{j \in [m]} \mu_j^k + \sum_{j \in [m]} \frac{(n_j)^2}{n}} \leq \frac{n^*}{\sum_{j \in [m]} \frac{(n_j)^2}{n}} \leq \frac{m}{2\sqrt{m-2}},$$

which proves the statement.  $\square$

## B.4 Welfare Approximation of PIECEWISEUNIFORM and LADDER

We show that PIECEWISEUNIFORM does not achieve the same welfare approximation as UTILPROP.

**Proposition 6.** *For every  $n \in \mathbb{N}$  with  $n \geq 4$  even, there are instances with  $n$  voters in which, if  $a$  is the outcome of PIECEWISEUNIFORM, it holds  $\frac{n+1}{3}w(a) = w(a^{OPT})$ .*

*Proof.* Let  $n \in \mathbb{N}$  be an even number with  $n \geq 4$ , and consider the following profile with  $m = \frac{n^2}{2} + 1$  alternatives: Voters  $i$  from 1 to  $\frac{n}{2}$  vote  $\frac{1}{n}$  on every alternative  $(i-1) \cdot n + \ell$  for  $\ell \in [n]$ ; voters  $i$  from  $\frac{n}{2} + 1$  to  $n$  vote 1 on alternative  $\frac{n^2}{2} + 1$ . It is easy to see that the optimal allocation allocates all budget on the last alternative, i.e.,  $a^{OPT} = (0, \dots, 0, 1)$ , with  $w(a^{OPT}) = \frac{n}{2}$ .

PIECEWISEUNIFORM reaches normalization at time  $t^* = \frac{n+3}{2(n+1)}$ , when the phantom positions are  $f_k(t^*) = \frac{(n-k)(2t^*-1)}{n} = \frac{2(n-k)}{n(n+1)}$  for  $k \geq \frac{n}{2}$ , which are the relevant phantoms for this profile and time of normalization. In particular, we have  $f_{n-1}(t^*) = \frac{2}{n(n+1)}$  and  $f_{\frac{n}{2}}(t^*) = \frac{1}{n+1}$ , so the allocation is  $a = (\frac{2}{n(n+1)}, \dots, \frac{2}{n(n+1)}, \frac{1}{n+1})$  with welfare  $w(a) = \frac{n^2}{2} \cdot \frac{2}{n(n+1)} + \frac{n}{2} \cdot \frac{1}{n+1} = \frac{3n}{2(n+1)}$ . Since  $\frac{w(a^{OPT})}{w(a)} = \frac{n+1}{3}$ , the result follows.  $\square$

While we do not have a similar result for LADDER, we can show that LADDER does not satisfy the proportional spending property

**Proposition 7.** *LADDER does not satisfy proportional spending.*

*Proof.* Consider the profile  $P \in \mathcal{P}_{2,3}$  with  $n = 2$  voters and  $m = 3$  alternatives. Voter 1 votes  $p_1 = (\frac{5}{6}, \frac{1}{6}, 0)$  and voter 2 votes  $p_2 = (\frac{5}{6}, 0, \frac{1}{6})$ . LADDER reaches normalization at time  $t = \frac{4}{6}$ , when the phantoms are at positions  $f_0(\frac{4}{6}) = \frac{4}{6}$ ,  $f_1(\frac{4}{6}) = \frac{1}{6}$ , and  $f_2(\frac{4}{6}) = 0$ . The output allocation of LADDER is thus  $a = (\frac{4}{6}, \frac{1}{6}, \frac{1}{6})$ , which violates proportional spending for  $k = 1$ , since  $\square(a, \mu^k) = \frac{4}{6} < \frac{5}{6} = \min(1, \frac{5}{6}) = \min(\frac{n-k+1}{n}, \sum_{j \in [m]} \mu_j^k)$ .  $\square$

## C Omitted Material from Section 4

### C.1 Proof of Lemma 4

**Lemma 4.** *Let  $n \in \mathbb{N}$  be fixed,  $P = \{p_1, \dots, p_n\}$  be a profile and  $f_0, \dots, f_n \in [0, 1]$  and  $g_0, \dots, g_n \in [0, 1]$  the positions of the phantoms of two phantom systems  $\mathcal{F}_n$  and  $\mathcal{G}_n$  at a time of normalization (respectively) for  $P$ , i.e.,*

$$\sum_{j \in [m]} \text{med}(f_0, \dots, f_n, p_{1,j}, \dots, p_{n,j}) = \sum_{j \in [m]} \text{med}(g_0, \dots, g_n, p_{1,j}, \dots, p_{n,j}) = 1.$$

*Let  $a^{\mathcal{F}}$  and  $a^{\mathcal{G}}$  be the output allocations of the corresponding moving-phantom mechanisms. If there exists an index  $k^* \in \{-1, 0, \dots, n\}$  such that  $f_i \geq g_i$  for all  $i \in \{0, \dots, k^*\}$  and  $f_i \leq g_i$  for all  $i \in \{k^* + 1, \dots, n\}$ , then  $w(a^{\mathcal{F}}) \geq w(a^{\mathcal{G}})$ .*

*Proof.* We show that the increase in welfare due to the alternatives  $j$  receiving more from  $\mathcal{F}$  than from  $\mathcal{G}$  (i.e.,  $a_j^{\mathcal{F}} - a_j^{\mathcal{G}} > 0$ ) is larger than the decrease from alternatives receiving less. If  $k^* = -1$ , then all phantoms of  $\mathcal{F}_n$  are less than or equal to the phantoms from  $\mathcal{G}_n$ , implying that each median must be lower or equal as well. Due to normalization, we then get  $a^{\mathcal{F}} = a^{\mathcal{G}}$ . The same holds for  $k^* = n$ , thus we assume  $0 \leq k^* \leq n - 1$  from this point.

As both  $a^{\mathcal{F}}$  and  $a^{\mathcal{G}}$  are normalized, we have  $\sum_{j \in [m]} (a_j^{\mathcal{F}} - a_j^{\mathcal{G}}) = 0$ . Let  $M^+ = \{j \in [m] \mid a_j^{\mathcal{F}} > a_j^{\mathcal{G}}\}$  and  $M^- = \{j \in [m] \mid a_j^{\mathcal{F}} < a_j^{\mathcal{G}}\}$  denote the alternatives with a positive and a negative difference, respectively. We claim that, on any alternative  $j \in M^+$ , there are at least  $n - k^*$  voters with a vote larger than or equal to  $a_j^{\mathcal{F}}$ .

*Proof of claim:* Suppose there are  $k < n - k^*$  such voters. Then, by the definition of the median, there must be at least  $n + 1 - k > k^* + 1$  phantoms  $f_i$  at positions larger than or equal to  $a_j^{\mathcal{F}}$ , thus  $f_{n-k} \geq a_j^{\mathcal{F}}$ . But since  $n - k > k^*$ , we also know that  $f_{n-k} \leq g_{n-k}$  and thus  $g_{n-k} \geq a_j^{\mathcal{F}}$ . Therefore, there are  $k$  voters and at least  $n - k + 1$  phantoms  $g_i$  at positions larger than or equal to  $a_j^{\mathcal{F}}$ , contradicting the assumption  $a_j^{\mathcal{F}} > a_j^{\mathcal{G}}$ .  $\blacksquare$

Similarly, we claim that on each alternative  $j \in M^-$ , there are at most  $n - k^* - 1$  voters with a vote strictly larger than  $a_j^{\mathcal{F}}$ .

*Proof of claim:* Suppose there are  $k \geq n - k^*$  such voters. Then by the definition of the median, there can be only  $n - k \leq k^*$  phantoms  $f_i$  at positions strictly larger than  $a_j^{\mathcal{F}}$ , thus  $f_{n-k} \leq a_j^{\mathcal{F}}$ . But since  $n - k \leq k^*$ , we also know that  $f_{n-k} \geq g_{n-k}$  and thus  $g_{n-k} \leq a_j^{\mathcal{F}}$ . Therefore, there are  $n - k$  voters and at least  $k + 1$  phantoms  $g_i$  at positions lower than or equal to  $a_j^{\mathcal{F}}$ , contradicting the assumption  $a_j^{\mathcal{F}} < a_j^{\mathcal{G}}$ .  $\blacksquare$

Combining the previous claims, we obtain

$$\begin{aligned}
& w(a^{\mathcal{F}}) - w(a^{\mathcal{G}}) \\
&= \sum_{i \in [n]} \sum_{j \in [m]} \min(p_{i,j}, a_j^{\mathcal{F}}) - \sum_{i \in [n]} \sum_{j \in [m]} \min(p_{i,j}, a_j^{\mathcal{G}}) \\
&= \sum_{j \in M^+} \sum_{i \in [n]} \underbrace{(\min(p_{i,j}, a_j^{\mathcal{F}}) - \min(p_{i,j}, a_j^{\mathcal{G}}))}_{\geq (n-k^*)(a_j^{\mathcal{F}} - a_j^{\mathcal{G}})} - \sum_{j \in M^-} \sum_{i \in [n]} \underbrace{(\min(p_{i,j}, a_j^{\mathcal{G}}) - \min(p_{i,j}, a_j^{\mathcal{F}}))}_{< (n-k^*)(a_j^{\mathcal{G}} - a_j^{\mathcal{F}})} \\
&> (n - k^*) \underbrace{\left( \sum_{j \in M^+} (a_j^{\mathcal{F}} - a_j^{\mathcal{G}}) - \sum_{j \in M^-} (a_j^{\mathcal{G}} - a_j^{\mathcal{F}}) \right)}_{=0} \\
&= 0. \tag*{$\square$}
\end{aligned}$$

## C.2 Proof of Theorem 3

**Theorem 3.** *The following relations hold:*

- (i) *UTIL is welfare-maximizing and CONSTANT is welfare-minimizing among moving-phantom mechanisms.*
- (ii) *UTILPROP is welfare-maximizing and FAN is welfare-minimizing among single-minded proportional moving-phantom mechanisms.*
- (iii) *PIECEWISEUNIFORM  $\supseteq$  INDEPENDENTMARKETS and LADDER  $\supseteq$  INDEPENDENTMARKETS, but PIECEWISEUNIFORM and LADDER are incomparable according to  $\supseteq$ .*
- (iv) *FAN  $\supseteq$  GREEDYMAX.*

*Proof of Theorem 3 (i).* It is well known that UTIL is welfare-maximizing among moving-phantom mechanisms [Freeman et al., 2021]. To see that CONSTANT is welfare-minimizing, we observe that for any profile, CONSTANT attains normalization when all phantoms are at position  $\frac{1}{m}$ . Let  $n, m$  be fixed,  $P \in \mathcal{P}_{n,m}$  be a profile, and  $\mathcal{F}_n$  a phantom system attaining normalization at time  $t$ . Choose  $k^*$  to be the index of the lowest phantom at a position  $\frac{1}{m}$  or larger, or  $k^* = -1$  if no such phantom exists. Then, we have  $f_i(t) \geq \frac{1}{m}$  for all  $i \in \{0, \dots, k^*\}$  and  $f_i \leq \frac{1}{m}$  for all  $i \in \{k^* + 1, \dots, n\}$ . By Lemma 4, we conclude that  $\mathcal{A}^{\mathcal{F}_n} \supseteq \text{CONSTANT}$ .  $\square$

*Proof of Theorem 3 (ii) (second part).* Fix  $n \in \mathbb{N}$  and let  $\mathcal{G}_n$  be the phantom system of FAN. By Lemma 5, for any single-minded proportional moving-phantom mechanism  $\mathcal{A}$ , we can find a phantom system  $\mathcal{F}_n = \{f_k \mid k \in [n]_0\}$  inducing it such that  $f_k(1) = 1 - \frac{k}{n}$  for each  $k \in [n]_0$ .

We show that for any two times  $t, t' \in [0, 1]$ , we can find a threshold index  $k^*$  such that all phantoms with index  $k \leq k^*$  from  $\mathcal{F}_n$  lie above the ones from  $\mathcal{G}_n$  at times  $t$  and  $t'$ , respectively, and all the phantoms with index  $k > k^*$  lie below. Indeed, let  $k^*$  be the highest index such that  $f_{k^*}(t) \geq t'$ , or choose  $k^* = -1$  if such an index does not exist. Note that, by the definition of FAN, we then have  $f_k(t) \geq t' \geq g_k(t')$  for all  $k \leq k^*$ . For the lower phantoms of  $\mathcal{F}_n$  with  $k > k^*$ , we know by monotonicity that they are lower than (or equal to) both  $t'$  and their end positions, i.e.,  $f_k(t) \leq \min(t', 1 - \frac{k}{n}) = g_k(t')$  for all  $k > k^*$ . Applying Lemma 4, with  $t$  and  $t'$  equal to the respective times of normalization, yields  $\mathcal{A} \supseteq \text{FAN}$ .  $\square$

*Proof of Theorem 3 (iii).* We first prove that  $\text{PIECEWISEUNIFORM} \supseteq \text{INDEPENDENTMARKETS}$ . Let  $\mathcal{F} = \{\mathcal{F}_n \mid n \in \mathbb{N}\}$  and  $\mathcal{G} = \{\mathcal{G}_n \mid n \in \mathbb{N}\}$  be the families of phantom systems of  $\text{PIECEWISEUNIFORM}$  and  $\text{INDEPENDENTMARKETS}$ , respectively. Fix some  $n \in \mathbb{N}$ . We show that for any two times  $t, t' \in [0, 1]$ , we can find a threshold index  $k^*$  such that  $f_i(t) \geq g_i(t')$  for all  $i \in \{0, \dots, k^*\}$  and  $f_i(t) \leq g_i(t')$  for all  $i \in \{k^* + 1, \dots, n\}$ . Applying Lemma 4 then completes the proof.

Note that at time  $t'$ , the phantoms of  $\text{INDEPENDENTMARKETS}$  are spaced equally with a distance of  $d^{\text{IM}} = \frac{t'}{n}$  between them. Similarly, the lower phantoms  $f_n$  to  $f_{\lfloor \frac{n}{2} \rfloor + 1}$  of  $\text{PIECEWISEUNIFORM}$  are equally spaced with a distance of  $d_\ell^{\text{PU}} = \max\left(0, \frac{2t-1}{n}\right)$  between them. The upper phantoms  $f_{\lfloor \frac{n}{2} \rfloor}$  to  $f_0$  are also equally spaced, with a distance  $d_u^{\text{PU}}$  of  $\frac{4t}{n}$  if  $t < \frac{1}{2}$  and  $\frac{3-2t}{n}$  if  $t \geq \frac{1}{2}$ .

We scan the phantoms from  $\text{PIECEWISEUNIFORM}$  from bottom to top and compare their positions with the phantoms from  $\text{INDEPENDENTMARKETS}$ . Both lowest phantoms are at position 0, i.e.  $f_n(t) = 0 = g_n(t')$ . If  $d_\ell^{\text{PU}} = \max\left(0, \frac{2t-1}{n}\right) > \frac{t'}{n} = d^{\text{IM}}$ , then we have  $t > \frac{1}{2}$  and thus  $d_u^{\text{PU}} = \frac{3-2t}{n} \geq \frac{1}{n} \geq d^{\text{IM}}$ . Therefore, all phantoms of  $\text{PIECEWISEUNIFORM}$  are above or equal to the phantoms from  $\text{INDEPENDENTMARKETS}$ , and we can choose  $k^* = n$ . Otherwise ( $d_\ell^{\text{PU}} \leq d^{\text{IM}}$ ), we know that all lower phantoms from  $\text{PIECEWISEUNIFORM}$  are at most at the phantom positions from  $\text{INDEPENDENTMARKETS}$ , i.e.,  $f_k(t) \leq g_k(t')$  for all  $k \in \{\lfloor \frac{n}{2} \rfloor + 1, \dots, n\}$ . If  $d_u^{\text{PU}} \leq d^{\text{IM}}$ , then the same holds for the remaining phantoms, and we can choose  $k^* = -1$ . Otherwise, choose  $k^* \leq \lfloor \frac{n}{2} \rfloor$  to be the index of the lowest phantom with  $f_{k^*}(t) > g_{k^*}(t')$  or  $k^* = -1$  if no such index exists. Then, for all phantoms  $k < k^*$  from  $\text{PIECEWISEUNIFORM}$  above  $f_{k^*}$ , we have

$$f_k(t) = f_{k^*}(t) + (k^* - k)d_u^{\text{PU}} > g_{k^*}(t') + (k^* - k)d^{\text{IM}} = g_k(t').$$

We next prove that  $\text{LADDER} \supseteq \text{INDEPENDENTMARKETS}$ . Let  $\mathcal{F} = \{\mathcal{F}_n \mid n \in \mathbb{N}\}$  and  $\mathcal{G} = \{\mathcal{G}_n \mid n \in \mathbb{N}\}$  be the families of phantom systems of  $\text{LADDER}$  and  $\text{INDEPENDENTMARKETS}$ , respectively. Fix some  $n \in \mathbb{N}$ . We show that for any two times  $t, t' \in [0, 1]$ , we can find a threshold index  $k^*$  such that  $f_i(t) \geq g_i(t')$  for all  $i \in \{0, \dots, k^*\}$  and  $f_i(t) \leq g_i(t')$  for all  $i \in \{k^* + 1, \dots, n\}$ . Applying Lemma 4 then completes the proof.

We choose  $k^*$  to be the largest index such that  $f_{k^*}(t) > g_{k^*}(t')$ . If this does not exist, then we can choose  $k^* = -1$  and conclude. Note that, in particular, we have  $f_{k^*}(t) > 0$ , since  $g_k(t') \geq 0$  for any  $k$ . By the definition of  $\text{LADDER}$  and  $\text{INDEPENDENTMARKETS}$ , we have for each  $k < k^*$

$$\begin{aligned} f_k(t) &= t - \frac{k}{n} = t - \frac{k^*}{n} + \frac{k^* - k}{n} = f_{k^*}(t) + \frac{k^* - k}{n} \\ &> g_{k^*}(t') + t' \cdot \frac{k^* - k}{n} = t' \cdot \frac{n - k^*}{n} + t' \cdot \frac{k^* - k}{n} = t' \cdot \frac{n - k}{n} = g_k(t'). \end{aligned}$$

Similarly, for each  $k \geq k^* + 1$  with  $f_k(t) > 0$  we know that

$$\begin{aligned} f_k(t) &= t - \frac{k}{n} = t - \frac{k^* + 1}{n} - \frac{k - k^* - 1}{n} = f_{k^*+1}(t) - \frac{k - k^* - 1}{n} \\ &\leq g_{k^*+1}(t') - t' \cdot \frac{k - k^* - 1}{n} = t' \cdot \frac{n - k^* - 1}{n} - t' \cdot \frac{k - k^* - 1}{n} = t' \cdot \frac{n - k}{n} = g_k(t'). \end{aligned}$$

Finally, for all  $k > k^*$  with  $f_k(t) = 0$  it is trivial that  $f_k(t) = 0 \leq g_k(t')$ .

For the last part of the claim in the statement, we construct a profile  $P$  where  $\text{LADDER}$  achieves higher welfare than  $\text{PIECEWISEUNIFORM}$ , and a profile  $P'$  for which the opposite relation holds.

Let  $n = 4$  and  $m = 3$ . Profiles  $P, P' \in \mathcal{P}_{4,3}$  are defined by the votes

$$\begin{array}{ll} p_1 = (1, 0, 0) & p'_1 = (\frac{1}{2}, \frac{1}{2}, 0) \\ p_2 = (0, 1, 0) & p'_2 = (\frac{1}{2}, \frac{1}{2}, 0) \\ p_3 = (0, 0, 1) & p'_3 = (\frac{1}{2}, 0, \frac{1}{2}) \\ p_4 = (\frac{1}{2}, \frac{1}{2}, 0) & p'_4 = (0, \frac{1}{2}, \frac{1}{2}) \end{array}$$

In profile  $P$ , LADDER reaches normalization at time  $t = \frac{11}{12}$  when the phantom positions are  $(\frac{11}{12}, \frac{2}{3}, \frac{5}{12}, \frac{1}{6}, 0)$ , outputting the allocation  $(\frac{5}{12}, \frac{5}{12}, \frac{1}{6})$  with welfare  $\frac{11}{6}$ . PIECEWISEUNIFORM reaches normalization at time  $t = \frac{9}{10}$  when the phantom positions are  $(1, \frac{7}{10}, \frac{2}{5}, \frac{1}{5}, 0)$ , outputting the allocation  $(\frac{2}{5}, \frac{2}{5}, \frac{1}{5})$  with welfare  $\frac{9}{5} < \frac{11}{6}$ .

In profile  $P'$ , LADDER reaches normalization at time  $t = \frac{2}{3}$  when the phantom positions are  $(\frac{2}{3}, \frac{5}{12}, \frac{1}{6}, 0, 0)$ , outputting the allocation  $(\frac{5}{12}, \frac{5}{12}, \frac{1}{6})$  with welfare  $\frac{17}{6}$ . PIECEWISEUNIFORM reaches normalization at time  $t = \frac{1}{2}$  when the phantom positions are  $(1, \frac{1}{2}, 0, 0, 0)$ , outputting the allocation  $(\frac{1}{2}, \frac{1}{2}, 0)$  with welfare  $3 > \frac{17}{6}$ .  $\square$

*Proof of Theorem 3 (iv).* Let  $\mathcal{F} = \{\mathcal{F}_n \mid n \in \mathbb{N}\}$  and  $\mathcal{G} = \{\mathcal{G}_n \mid n \in \mathbb{N}\}$  be the families of phantom systems of FAN and GREEDYMAX, respectively. Fix some  $n \in \mathbb{N}$ . We show that for any two times  $t, t' \in [0, 1]$ , we can find a threshold index  $k^*$ , such that  $f_i(t) \geq g_i(t')$  for all  $i \in \{0, \dots, k^*\}$  and  $f_i(t) \leq g_i(t')$  for all  $i \in \{k^* + 1, \dots, n\}$ . Applying Lemma 4 then completes the proof.

We choose  $k^*$  to be the highest index such that  $f_{k^*}(t) \geq g_0(t')$ , or  $k^* = -1$  if no such index exists. If  $k^* = n$ , then  $t' = 0$ , since  $f_n(t) = 0$  and thus  $f_i(t) \geq g_i(t') = 0$  for all  $i = 0, \dots, k^* = n$  follows trivially. Otherwise, we have  $f_k(t) \geq g_k(t') = g_0(t')$  for all  $k = 0, \dots, k^*$  and  $f_k(t) \leq g_k(t') = g_0(t')$  for all  $k \in \{k^* + 1, \dots, n - 1\}$ . Since  $f_n(t) = 0 \leq 0 = g_n(t')$  also holds, we conclude.  $\square$

### C.3 Proof of Lemma 5

**Lemma 5.** *A moving-phantom mechanism  $\mathcal{A}$  is single-minded proportional if and only if there is a family of phantom systems,  $\mathcal{F}_n = \{f_k \mid k \in [n]_0\}$  for each  $n \in \mathbb{N}$ , that produces the same outcome as  $\mathcal{A}$  for every instance, with the property that for all  $n \in \mathbb{N}$  we have  $f_k(1) = \frac{n-k}{n}$  for each  $k \in [n]_0$ .*

*Proof.* The backwards direction of the implication was shown by Freeman et al. [2021]; we include it here for completeness. Let  $n$  be fixed,  $P \in \mathcal{P}_{n,m}$  be a single-minded profile, and  $n_j$  denote the number of voters voting 1 on alternative  $j \in [m]$ . Then, at time  $t = 1$ , the median on each alternative  $j$  is the  $(n - n_j)^{\text{th}}$  phantom, which is at position  $f_{n-n_j}(1) = 1 - \frac{n-n_j}{n} = \frac{n_j}{n} = \frac{1}{n} \sum_{i \in [n]} p_{i,j}$ . Thus, the mechanism is single-minded proportional.

For the forward direction, let  $\mathcal{A}^{\mathcal{G}}$  be a single-minded proportional moving-phantom mechanism induced by a phantom system  $\mathcal{G}_n = \{g_k \mid k \in [n]_0\}$  for each  $n \in \mathbb{N}$ . Recall that, by the definition of moving-phantom mechanisms,  $g_k(1) \geq \frac{n-k}{n}$  for all  $k \in [n]_0$ .

We consider the phantom system  $\mathcal{F}_n = \{f_k(t) = \min(g_k(t), \frac{n-k}{n}) \mid k \in [n]_0\}$ , and claim that for any profile  $P$ , any time  $t$  of normalization for  $\mathcal{G}_n$  is also a time of normalization for  $\mathcal{F}_n$ . Since all phantoms and thus medians of  $\mathcal{F}_n$  are at most as high as the ones of  $\mathcal{G}_n$ , normalization then implies that  $\mathcal{A}^{\mathcal{G}}(P) = \mathcal{A}^{\mathcal{F}}(P)$ , concluding the proof of the theorem.

We fix  $n \in \mathbb{N}$  in what follows. Before showing the claim, we make the following observation: For  $k \in [n]_0$  and  $t \in [0, 1]$ ,

$$g_k(t) > 1 - \frac{k}{n} \implies g_{k'}(t) \geq 1 - \frac{k'}{n} \text{ for all } k' \in [n]_0 \text{ with } k' \geq n - k. \quad (6)$$

Indeed, suppose that  $g_k(t) > 1 - \frac{k}{n}$  for some  $k \in [n]_0$  and  $t \in [0, 1]$  and that there is some  $k' \geq n - k$  with  $g_{k'}(t) < 1 - \frac{k'}{n}$ . Consider the following profile in  $\mathcal{P}_{n,3}$ , where  $k \times p$  denotes  $k$  voters voting  $p$ :

$$((n - k) \times (1, 0, 0), (n - k') \times (0, 1, 0), (k + k' - n) \times (0, 0, 1)).$$

For  $\mathcal{A}^{\mathcal{G}}$  to be single-minded proportional, there must be a time  $t'$  at which  $g_k(t') = 1 - \frac{k}{n}$  and  $g_{k'}(t') = 1 - \frac{k'}{n}$ . But then  $g_k(t') < g_k(t)$  and  $g_{k'}(t') > g_{k'}(t)$ , contradicting the monotonicity of the phantom functions.

We now prove the claim from above. Let  $P \in \mathcal{P}_{n,m}$  be a profile and  $t \in [0, 1]$  a time of normalization for the phantom system  $\mathcal{G}_n$ . We will iteratively move the phantoms from  $\mathcal{G}_n$  continuously down to the positions of the phantoms from  $\mathcal{F}_n$ , starting with the lowest phantom  $g_n$ . If none of the medians change during this process, then  $t$  is a time of normalization for  $\mathcal{F}$  and we conclude. Otherwise, we let  $k^*$  be the index of the first (lowest) phantom whose movement changes the median, and  $j$  be an alternative where the median changes. Since the median is continuous and always equal to one of its input values, when moving the phantom from  $g_{k^*}(t)$  to  $f_{k^*}(t)$  there must be a position  $\tau \in (f_{k^*}(t), g_{k^*}(t)) \setminus \{p_{i,j} \mid i \in [n]\}$  at which the median is exactly  $\tau$ . We denote the adjusted positions of the phantoms as  $h_k \in [f_k(t), g_k(t)]$  for each  $k \in [n]_0$ , i.e.,  $h_k = f_k(t)$  for  $k > k^*$ ,  $h_{k^*} = \tau$ , and  $h_k = g_k(t)$  for  $k < k^*$ .

Now, let  $n_j = n - k^*$  be the number of voters strictly above  $h_{k^*} = \tau$  on alternative  $j$ . We manipulate the profile, without changing the median on  $j$ , by making these  $n_j$  voters single-minded on  $j$ . Note that both moving the phantoms down and making the voters single-minded can only decrease the medians on all alternatives. Since no voter votes exactly  $\tau$  on  $j$ , each of the remaining  $n - n_j$  voters votes strictly below  $\tau$  on  $j$  and has some alternative  $j' \neq j$  with a vote strictly higher than the current median by the normalization of the votes. We make these voters single-minded on these alternatives to obtain a single-minded profile, which again can not increase any medians. Moreover, the median on alternative  $j$  is still  $h_{k^*} = \tau > f_{k^*}(t) = \frac{n_j}{n}$ . By Observation (6), from  $g_{k^*}(t) = g_{n-n_j}(t) > \frac{n_j}{n}$  it follows that  $g_k(t) \geq 1 - \frac{k}{n}$  for  $k \geq n_j$ . Let  $n_{j'}$  denote the number of voters now voting 1 on each alternative  $j' \neq j$ . Then, on each alternative  $j' \neq j$ , we have  $n - n_{j'} \geq n_j$  and thus the median on  $j'$  is exactly

$$h_{n-n_{j'}} \geq f_{n-n_{j'}}(t) = \min \left( g_{n-n_{j'}}(t), \frac{n_{j'}}{n} \right) = \frac{n_{j'}}{n}.$$

But then, the current sum of medians is

$$h_{k^*} + \sum_{j' \in [m]: j' \neq j} h_{n-n_{j'}} > \frac{n_j}{n} + \sum_{j' \in [m]: j' \neq j} \frac{n_{j'}}{n} = 1,$$

contradicting the normalization of  $\mathcal{G}_n$  at time  $t$ . □

#### C.4 Lattice of Phantom Mechanisms for Two Alternatives

**Proposition 8.** *For  $m = 2$  alternatives, the dominance relation  $\succeq$  among moving-phantom mechanisms forms a lattice.*

*Proof.* For 2 alternatives, by Freeman et al. [2021, Theorem 5] we can assume w.l.o.g. that every moving-phantom mechanism is induced by a *symmetric generalized median mechanism*. More precisely, this means that the moving-phantom mechanism can be expressed by a phantom systems  $\mathcal{F}_n = \{f_k \mid k = 0, \dots, n\}$  satisfying the following two properties: (i) phantoms are constant, i.e., for all  $k \in [n]_0$  there exists  $f_k \in [0, 1]$  with  $f_k(t) = f_k$  for all  $t \in [0, 1]$ , and (ii)  $f_k = 1 - f_{n-k}$  for every  $k \in \{0, \dots, \lfloor \frac{n}{2} \rfloor\}$ .

We claim that, for two families of symmetric phantom systems  $\mathcal{F} = \{\mathcal{F}_n \mid n \in \mathbb{N}\}$  and  $\mathcal{G} = \{\mathcal{G}_n \mid n \in \mathbb{N}\}$  given by  $\mathcal{F}_n = \{f_k \mid k = 0, \dots, \lfloor \frac{n}{2} \rfloor\}$  and  $\mathcal{G}_n = \{g_k \mid k = 0, \dots, \lfloor \frac{n}{2} \rfloor\}$  for each  $n \in \mathbb{N}$ , it holds that

$$\mathcal{A}^{\mathcal{F}} \supseteq \mathcal{A}^{\mathcal{G}} \iff f_k \geq g_k \text{ for all } k \in \left\{0, \dots, \left\lfloor \frac{n}{2} \right\rfloor\right\}, n \in \mathbb{N}.$$

If the right-hand side of this equivalence is true, then we can apply Lemma 4 with  $k^* = \lfloor \frac{n}{2} \rfloor$ , and conclude that  $\mathcal{A}^{\mathcal{F}} \supseteq \mathcal{A}^{\mathcal{G}}$ . If the right-hand side is not true, there exists  $n \in \mathbb{N}$  and  $k \in \{0, \dots, \lfloor \frac{n}{2} \rfloor\}$  such that  $f_k < g_k$ . Note that, when  $n$  is even,  $f_{\frac{n}{2}} = g_{\frac{n}{2}} = \frac{1}{2}$  by symmetry, so we have  $k < \frac{n}{2}$ . Consider the profile  $P \in \mathcal{P}_{n,2}$  where the voters  $i$  from 1 to  $n-k$  vote  $(1, 0)$  and the voters  $n-k+1$  to  $n$  vote  $(0, 1)$ . Then, the outcomes are  $\mathcal{A}^{\mathcal{F}}(P) = (f_k, 1 - f_k)$  and  $\mathcal{A}^{\mathcal{G}}(P) = (g_k, 1 - g_k)$ , with respective welfares of

$$w(\mathcal{A}^{\mathcal{F}}) = (n-k)f_k + k(1-f_k) = k + (n-2k)f_k < k + (n-2k)g_k = (n-k)g_k + k(1-g_k) = w(\mathcal{A}^{\mathcal{G}}),$$

where the inequality follows from the facts that  $k < \frac{n}{2}$  and  $f_k < g_k$ .

Having established the claim, for any two families of symmetric phantom systems  $\mathcal{F} = \{\mathcal{F}_n \mid n \in \mathbb{N}\}$  and  $\mathcal{G} = \{\mathcal{G}_n \mid n \in \mathbb{N}\}$  given by  $\mathcal{F}_n = \{f_k \mid k = 0, \dots, \lfloor \frac{n}{2} \rfloor\}$  and  $\mathcal{G}_n = \{g_k \mid k = 0, \dots, \lfloor \frac{n}{2} \rfloor\}$  for each  $n \in \mathbb{N}$ , it follows that:

- (i) The least upper bound or *join* of their associated moving-phantom mechanisms according to  $\supseteq$  is given by  $\mathcal{A}^{\mathcal{F}} \vee \mathcal{A}^{\mathcal{G}} = \mathcal{A}^{\mathcal{H}}$ , where  $\mathcal{H} = \{\mathcal{H}_n \mid n \in \mathbb{N}\}$  is given, for each  $n \in \mathbb{N}$ , by the symmetric phantom system  $\mathcal{H}_n = \{\max\{f_k, g_k\} \mid k = 0, \dots, \lfloor \frac{n}{2} \rfloor\}$ ;
- (ii) The greatest lower bound or *meet* of their associated moving-phantom mechanisms according to  $\supseteq$  is given by  $\mathcal{A}^{\mathcal{F}} \wedge \mathcal{A}^{\mathcal{G}} = \mathcal{A}^{\mathcal{H}}$ , where  $\mathcal{H} = \{\mathcal{H}_n \mid n \in \mathbb{N}\}$  is given, for each  $n \in \mathbb{N}$ , by the symmetric phantom system  $\mathcal{H}_n = \{\min\{f_k, g_k\} \mid k = 0, \dots, \lfloor \frac{n}{2} \rfloor\}$ .  $\square$

## D Omitted Material from Section 5

### D.1 Pareto Suboptimality of GREEDYDECOMP and UTILDECOMP

**Proposition 9.** *Neither GREEDYDECOMP nor UTILDECOMP is Pareto-optimal.*

*Proof.* Consider the profile  $P$  with  $n = 7$  and  $m = 4$  defined as

$$P = \begin{pmatrix} 3/7 & 0 & 4/7 & 0 \\ 0 & 3/7 & 4/7 & 0 \\ 0 & 0 & 1 & 0 \\ 2/7 & 0 & 0 & 5/7 \\ 2/7 & 0 & 0 & 5/7 \\ 0 & 2/7 & 0 & 5/7 \\ 0 & 2/7 & 0 & 5/7 \end{pmatrix}.$$

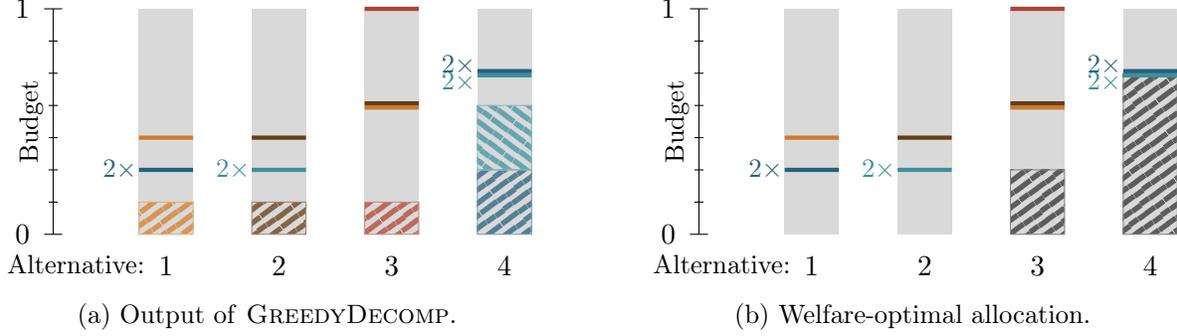


Figure 8: Visualization of the profile and aggregates from the proof of Proposition 9

Figure 8 illustrates this profile.

GREEDYDECOMP first exhausts the budget of voters 4 to 7 on alternative 4, and then spends the budget of voters 1 to 3 on alternatives 1 to 3, respectively. This leads to the allocation  $(\frac{1}{7}, \frac{1}{7}, \frac{1}{7}, \frac{4}{7})$  (see Figure 8a). Note that this is also an optimal decomposable allocation for this profile, as no decomposable mechanism can spend more than  $\frac{4}{7}$  on alternative 4. However, this outcome is Pareto-dominated by the non-decomposable allocation  $(0, 0, \frac{2}{7}, \frac{5}{7})$  (see Figure 8b), where all voters get as much utility as before, except for voter 3, whose utility strictly increases.  $\square$

## D.2 IP Formulation for Computing an Optimal Decomposable Allocation

For a fixed profile  $P = (p_{i,j})_{i \in [n], j \in [m]} \in \mathcal{P}_{n,m}$ , we consider the following integer linear program  $\text{IP}(P)$  with variables  $x \in \{0, 1\}^{n \times m}$  and  $c, u \in [0, 1]^{n \times m}$ :

$$\begin{aligned} \max \quad & \sum_{i \in [n]} \sum_{j \in [m]} u_{i,j} \\ & \sum_{j \in [m]} c_{i,j} = \frac{1}{n} && \text{for all } i \in [n], \end{aligned} \quad (7)$$

$$c_{i,j} \leq x_{i,j} \quad \text{for all } i \in [n], j \in [m], \quad (8)$$

$$\sum_{i' \in [n]} c_{i',j} + (1 - p_{i,j}) \cdot x_{i,j} \leq 1 \quad \text{for all } i \in [n], j \in [m], \quad (9)$$

$$u_{i,j} \leq \sum_{i' \in [n]} c_{i',j} \quad \text{for all } i \in [n], j \in [m], \quad (10)$$

$$u_{i,j} \leq p_{i,j} \quad \text{for all } i \in [n], j \in [m]. \quad (11)$$

**Proposition 10.** *For a profile  $P \in \mathcal{P}_{n,m}$  and  $c \in [0, 1]^{n \times m}$ ,  $a \in \Delta^{(m-1)}$  is a decomposable allocation with voter contributions  $c$  if and only if there exist  $x \in \{0, 1\}^{n \times m}$  and  $u \in [0, 1]^{n \times m}$  such that  $(x, c, u)$  is a feasible solution for  $\text{IP}(P)$ .*

*Proof.* We fix  $P \in \mathcal{P}_{n,m}$  and  $c \in [0, 1]^{n \times m}$ . For the forward direction, we let  $a \in \Delta^{(m-1)}$  be a decomposable allocation with voter contributions  $c \in [0, 1]^{n \times m}$ ; i.e.,  $\sum_{j \in [m]} c_{i,j} = \frac{1}{n}$  for all  $i \in [n]$ ,  $\sum_{i \in [n]} c_{i,j} = a_j$  for all  $j \in [m]$ , and  $a_j \leq p_{i,j}$  for all  $i \in [n], j \in [m]$  with  $c_{i,j} > 0$ . We define

$x \in \{0, 1\}^{n \times m}$  and  $u \in [0, 1]^{n \times m}$  as follows:

$$x_{i,j} = \begin{cases} 1 & \text{if } c_{i,j} > 0, \\ 0 & \text{otherwise,} \end{cases} \quad u_{i,j} = \min\{a_j, p_{i,j}\}, \quad \text{for all } i \in [n], j \in [m].$$

We claim that  $(x, c, u)$  is a feasible solution for  $\text{IP}(P)$ . That this solution satisfies constraints (7), (8), (10) and (11) is straightforward from its definition. For (9), we fix  $i \in [n]$  and  $j \in [m]$ , and distinguish two cases. If  $x_{i,j} = 1$ , the constraint becomes  $\sum_{i' \in [n]} c_{i',j} \leq p_{i,j}$ . Since  $c$  certifies decomposability, this is equivalent to  $a_j \leq p_{i,j}$  and holds true. If  $x_{i,j} = 0$ , the constraint becomes  $\sum_{i' \in [n]} c_{i',j} \leq 1$ , which is equivalent to  $a_j \leq 1$  and thus holds trivially by the feasibility of  $a$ .

For the backward direction, we let  $(x, c, u)$  be a feasible solution for  $\text{IP}(P)$  and consider  $a \in [0, 1]^m$  defined by  $a_j = \sum_{i \in [n]} c_{i,j}$  for every  $j \in [m]$ . From the fact that  $c \in [0, 1]^{n \times m}$  and constraint (7), it follows that  $a \in \Delta^{(m-1)}$ , i.e.,  $a$  is a feasible allocation. We prove in what follows that  $c$  are voter contributions that certify its decomposability. Two of the three conditions follow directly from constraint (7) and the definition of  $a$ : that  $\sum_{j \in [m]} c_{i,j} = \frac{1}{n}$  for all  $i \in [n]$ , and that  $\sum_{i \in [n]} c_{i,j} = a_j$  for all  $j \in [m]$ . It only remains to show that  $a_j \leq p_{i,j}$  for all  $i \in [n], j \in [m]$  with  $c_{i,j} > 0$ . Fix  $i \in [n]$  and  $j \in [m]$  such that  $c_{i,j} > 0$ . Constraint (8) implies  $x_{i,j} = 1$ , and thus constraint (9) yields  $p_{i,j} \geq \sum_{i' \in [n]} c_{i',j} = a_j$ , where the last equality follows from the definition of  $a$ .  $\square$

From our definition of voter utilities, captured precisely by  $\sum_{j \in [m]} u_{i,j}$  for each voter  $i$  due to constraints (10) and (11), we obtain the following corollary, stating an IP-based characterization of welfare-optimal decomposable allocations.

**Corollary 3.** *For any  $P \in \mathcal{P}_{n,m}$ ,  $a^* \in \Delta^{(m-1)}$  is a welfare-optimal decomposable allocation if and only if there exists an optimal solution  $(x^*, c^*, u^*)$  for  $\text{IP}(P)$  such that  $a_j^* = \sum_{i \in [n]} c_{i,j}^*$  for every  $j \in [m]$ .*

### D.3 Proof of Theorem 4

**Theorem 4.** *DECOMPOSABLEWELFARETHRESHOLD is NP-complete. In particular, computing the outcome of UTILDECOMP is NP-hard.*

We make use of the following problem for the reduction.

**Definition 2** (EXACTCOVERBY3SETS (X3C)). *An instance consists of a finite ground set  $U$  with  $|U| = 3q$  for some integer  $q \geq 1$ , and a family  $\mathcal{S} = \{S_1, \dots, S_k\} \subseteq \binom{U}{3}$  of 3-element subsets of  $U$ . The decision problem asks whether there exists a subfamily  $\mathcal{C} \subseteq \mathcal{S}$  of size  $q$  such that the sets in  $\mathcal{C}$  are pairwise disjoint and  $\bigcup_{S \in \mathcal{C}} S = U$  (i.e.,  $\mathcal{C}$  is an exact cover of  $U$ ).*

*Proof.* We reduce from EXACTCOVERBY3SETS (X3C). Note that the restricted version of X3C in which every element of  $U$  is contained in exactly three subsets, denoted RX3C, remains NP-complete [Gonzalez, 1985]. Given an instance  $(U, \mathcal{S})$  of RX3C we define the following budget aggregation instance. There are  $m = 3q + k + 2$  alternatives:  $k$  subset alternatives  $S_1, \dots, S_k$ ,  $3q$  dummy alternatives  $D_1, \dots, D_{3q}$ , one for each element, and two additional alternatives  $X$  and  $Y$ . Note that in the RX3C problem it is necessarily the case that  $k = 3q$ ; however, we will continue to treat  $k$  and  $q$  as distinct to make the mapping to elements and subsets more clear.

There are  $n = 21q + 2k \geq 21q \geq 21$  voters. We describe in the following the construction of a profile  $P \in \mathcal{P}_{n,m}$ . Let  $0 < \varepsilon < \frac{1}{n(q+1)}$  be a small positive constant. For every element  $i \in U$ , we consider

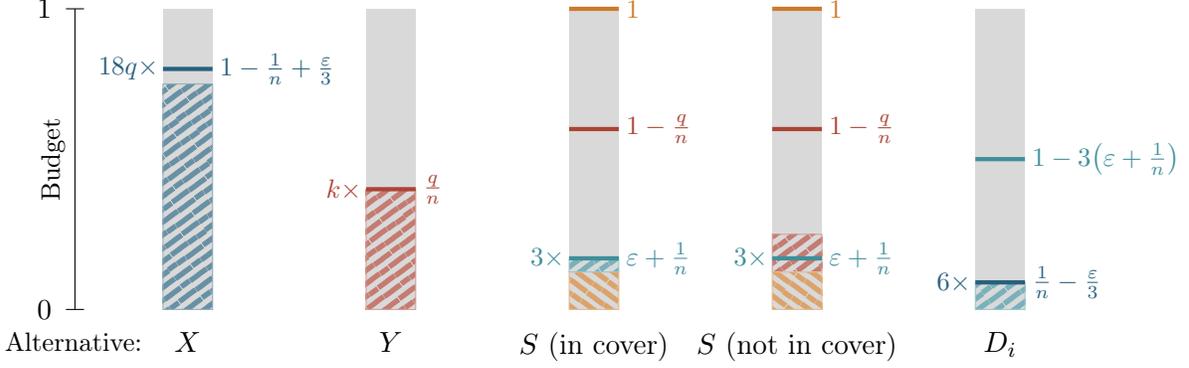


Figure 9: Illustration of the construction used in the proof of Theorem 4. Only non-zero votes are drawn and only representative examples of subset and dummy alternatives are shown. Blue line segments represent helper voters, orange line segments represent single-minded voters, red line segments represent toggle voters, and purple line segments represent element voters. Quantities to the left of the line segments indicate multiple voters at that position. Quantities to the right label the vote values. Colored hatching represents contributions of each voter type in the welfare-optimal decomposable allocation assuming that an exact cover exists. If an exact cover does not exist then the amount allocated to  $D_i$  increases while the amount allocated to subset alternatives decreases in the welfare-optimal decomposable allocation.

an *element* voter  $e_i$  with  $p_{e_i, S_j} = \varepsilon + \frac{1}{n}$  for each  $j$  with  $i \in S_j$  and  $p_{e_i, D_i} = 1 - 3(\varepsilon + \frac{1}{n})$ . For every subset  $S_i$ ,  $i \in [m]$ , we consider a *toggle* voter  $t_i$  with  $p_{t_i, Y} = \frac{q}{n}$  and  $p_{t_i, S_i} = 1 - \frac{q}{n}$ . Additionally, for every element  $i \in U$  we consider six additional *helper* voters  $h_{i,1}, \dots, h_{i,6}$  with  $p_{h_{i,\ell}, D_i} = \frac{1}{n} - \frac{\varepsilon}{3}$  and  $p_{h_{i,\ell}, X} = 1 - \frac{1}{n} + \frac{\varepsilon}{3}$ . Finally, for every subset  $S_i$ , we consider a single-minded voter  $s_i$  with  $p_{s_i, S_i} = 1$ . All other votes are zero.

We show that there exists a decomposable solution to the budget aggregation instance defined above with welfare at least  $\frac{324q^2 + kq + 28q + 4k}{n} + 4q\varepsilon$  if and only if the RX3C instance has an exact cover.

Suppose that there is an exact cover  $\mathcal{C}$  for the RX3C instance. We claim that the following decomposable allocation  $a^*$  achieves the required welfare: allocate  $a_X^* = \frac{18q}{n}$  to alternative  $X$ , allocate  $a_Y^* = \frac{q}{n}$  to alternative  $Y$ , allocate  $a_{D_i}^* = \frac{1}{n} - \frac{\varepsilon}{3}$  to each alternative  $D_i$ , allocate  $a_{S_i}^* = \frac{1}{n} + \varepsilon$  to every  $S_i \in \mathcal{C}$ , and allocate  $a_{S_i}^* = \frac{2}{n}$  to every  $S_i \notin \mathcal{C}$ .

To see that  $a^*$  is decomposable, we can define the following voter contributions. All helper voters  $h_{i,\ell}$ ,  $i \in [3q]$ ,  $\ell \in [6]$ , contribute all their  $1/n$  mass to alternative  $X$ , i.e.,  $c_{h_{i,\ell}, X} = 1/n$ . All single-minded voters  $s_i$  contribute all of their  $1/n$  mass to alternative  $S_i$ , i.e.,  $c_{s_i, S_i} = 1/n$ . For all  $i$  with  $S_i \in \mathcal{C}$ , toggle voters  $t_i$  contribute all  $1/n$  of their mass to alternative  $Y$ , i.e.,  $c_{t_i, Y} = 1/n$  (note that there are  $q$  such voters). For all  $i$  with  $S_i \notin \mathcal{C}$ , toggle voters  $t_i$  contribute all  $1/n$  of their mass to alternative  $S_i$ , i.e.,  $c_{t_i, S_i} = 1/n$  (note that there are  $k - q$  such voters). The element voters  $e_i$  contribute  $\varepsilon/3$  of their mass to the alternative  $S_j$  corresponding to the set that contains them in the exact cover and the remaining  $1/n - \varepsilon/3$  of their mass to alternative  $D_i$ , i.e.,  $c_{e_i, S_j} = \varepsilon/3$  for  $i \in S_j \in \mathcal{C}$  and  $c_{e_i, D_i} = 1/n - \varepsilon/3$ . Note that, by definition of an exact cover, there exists exactly one such  $S_j$  with  $i \in S_j \in \mathcal{C}$ . Notice that every voter contributes  $1/n$  in total and that the sum of contributions on each alternative is equal to that alternative's allocation. It remains to check that  $a_j^* \leq p_{i,j}$  for all voters  $i$  and all alternatives  $j$  with  $c_{i,j} > 0$ :

- Helper voters  $h_{i,\ell}$  contribute positively only to alternative  $X$ , and we have  $a_X^* = 18q/n \leq$

$$18/21 < 1 - \frac{1}{n} + \frac{\varepsilon}{3} = p_{h_{i,j},X}.$$

- Single-minded voters  $s_i$  contribute positively only to  $S_i$ , for which they have  $p_{s_i,S_i} = 1$ .
- For all  $i$  with  $S_i \in \mathcal{C}$ , toggle voters  $t_i$  contribute positively only to alternative  $Y$ , and we have  $a_Y^* = q/n = p_{t_i,Y}$ .
- For all  $i$  with  $S_i \notin \mathcal{C}$ , toggle voters  $t_i$  contribute positively only to alternative  $S_i$ , and we have  $a_{S_i}^* = 2/n < 20/21 \leq 1 - q/n = p_{t_i,S_i}$ .
- Element voters  $e_i$  contribute positively to an alternative  $S_j \in \mathcal{C}$ , for which  $a_{S_j}^* = \varepsilon + 1/n = p_{e_i,S_j}$ , and to alternative  $D_i$ , for which  $a_{D_i}^* = 1/n - \varepsilon/3 < 15/21 \leq 1 - 3(\varepsilon + 1/n) = p_{e_i,D_i}$ .

The utility of every helper voter is  $18q/n + 1/n - \varepsilon/3$ , of every element voter is  $3(\varepsilon + 1/n) + 1/n - \varepsilon/3$ , of every toggle voter  $t_i$  is  $q/n + \varepsilon + 1/n$  if  $S_i \in \mathcal{C}$  and  $q/n + 2/n$  if  $S_i \notin \mathcal{C}$ , and of every single-minded voter  $s_i$  is  $\varepsilon + 1/n$  if  $S_i \in \mathcal{C}$  and  $2/n$  if  $S_i \notin \mathcal{C}$ . Thus, the welfare of  $a^*$  is

$$\begin{aligned} w(a^*) &= 18q \left( \frac{18q}{n} + \frac{1}{n} - \frac{\varepsilon}{3} \right) + 3q \left( 3 \left( \varepsilon + \frac{1}{n} \right) + \frac{1}{n} - \frac{\varepsilon}{3} \right) + q \left( \frac{q}{n} + \varepsilon + \frac{1}{n} \right) + (k - q) \left( \frac{q}{n} + \frac{2}{n} \right) \\ &\quad + q \left( \varepsilon + \frac{1}{n} \right) + (k - q) \cdot \frac{2}{n} \\ &= \frac{324q^2 + kq + 28q + 4k}{n} + 4q\varepsilon \end{aligned}$$

We now show that on any RX3C instance for which there does not exist an exact cover, all decomposable budget allocations must achieve strictly lower welfare than  $\frac{324q^2 + kq + 28q + 4k}{n} + 4q\varepsilon$ . Note that, because decomposability implies that only  $q/n$  mass can be allocated to alternative  $Y$ , at least  $(k - q)/n$  mass from the toggle voters  $t_1, \dots, t_k$  must be allocated among subset alternatives  $S_1, \dots, S_k$ . Furthermore, at most  $1/n$  of the mass can be placed on each subset alternative (since no two toggle voters like the same one). In particular, this implies that at least  $k - q$  subset alternatives receive more than  $\varepsilon$  mass from toggle voters. To see this, suppose otherwise: that at most  $k - q - 1$  subset alternatives receive more than  $\varepsilon$  mass from toggle voters. Then the total amount of mass allocated among subset alternatives from toggle voters is at most

$$\frac{1}{n}(k - q - 1) + \varepsilon(q + 1) < \frac{k - q}{n} - \frac{1}{n} + \frac{1}{n} = \frac{k - q}{n},$$

where the inequality follows from  $\varepsilon < \frac{1}{n(q+1)}$ .

Note that every subset alternative  $S_j$  receives  $1/n$  mass from the corresponding single-minded voter  $s_j$  in any decomposable allocation. Thus, if alternative  $S_j$  additionally receives more than  $\varepsilon$  mass from toggle voter  $t_j$ , decomposability prohibits it from receiving positive mass from any element voter  $e_i$  such that  $i \in S_j$ , each of whom has  $p_{e_i,S_j} = 1/n + \varepsilon$ . Therefore, there are at most  $q$  subset alternatives that can receive mass from element voters. If there is no exact cover, then it is by definition the case that among any set of at most  $q$  subsets, there exists at least one element  $i$  not contained within any of the subsets. Therefore, the corresponding element voter  $e_i$  must allocate all of its  $1/n$  mass to alternative  $D_i$ .

Subject to allocating at most  $18q/n$  to alternative  $X$  (as required by decomposability) and exactly  $1/n$  to alternative  $D_i$  (as argued above), the welfare-optimal decomposable allocation  $a$  allocates  $18q/n$  to alternative  $X$ ,  $q/n$  to alternative  $Y$ ,  $1/n - \varepsilon/3$  to all alternatives  $D_j$  with  $j \neq i$ , and some amount  $x$  with  $1/n + \varepsilon \leq x \leq 2/n$  to each subset alternative  $S_j$ .

To justify optimality, observe that welfare can be maximized greedily by allocating budget to segments with the highest marginal contribution. Every unit of spending on  $X$  contributes to the utility of  $18q$  voters; every unit of spending up to  $q/n$  on  $Y$  contributes to the utility of  $k$  voters; every unit of spending up to  $1/n - \varepsilon/3$  on a dummy alternative contributes to the utility of 7 voters; and every unit of spending up to  $1/n + \varepsilon$  on a subset alternative contributes to the utility of 5 voters. The allocation therefore saturates all such regions subject to decomposability. Any remaining budget is optimally spent on subset alternatives in the range  $(1/n + \varepsilon, 1 - q/n]$ , where each unit contributes to the utility of only 2 voters. All other admissible marginal spending contributes to the utility of at most one voter and is therefore minimized, with the unavoidable exception of the mass allocated to  $D_i$ .

Computing the welfare of this allocation yields

$$w(a) = \frac{324q^2 + kq + 28q + 4k}{n} + \left(4q - \frac{1}{3}\right)\varepsilon < \frac{324q^2 + kq + 28q + 4k}{n} + 4q\varepsilon.$$

In particular,  $w(a) = w(a^*) - \varepsilon/3$ , reflecting the fact that an amount  $\varepsilon/3$  of budget is shifted from a segment of a subset alternative approved by two voters to a dummy alternative approved by only one voter.  $\square$

#### D.4 Proof of Theorem 5

**Theorem 5.** *For any profile with  $n \geq 2$  voters, UTILDECOMP achieves at most  $2 - \frac{1}{n-1}$  times the welfare that GREEDYDECOMP achieves.*

For this proof, we introduce some additional notation. We define an *allocation region*  $A \subseteq [0, 1]^m$  to be a vector of subsets, where each subset  $A_j$  is a disjoint union of intervals. An allocation  $a$  then corresponds to the allocation region  $A$  with  $A_j = [0, a_j)$ . We define the intersection and exclusion of two allocation regions  $A$  and  $A'$  as  $A \cap A' = (A_j \cap A'_j)_{j \in [m]}$  and  $A \setminus A' = (A_j \setminus A'_j)_{j \in [m]}$ . Even though allocation regions that are not of the form  $[0, x)$  for each alternative do not have an intuitive interpretation, they can be a useful tool for computing the welfare of an allocation. To this end, for a disjoint union of  $x$  intervals  $X = \bigcup_{k=1}^x X_k$ , we let  $|X|$  denote the Lebesgue measure of  $X$ , which corresponds to the sum of the lengths of the sub-intervals, i.e.,  $|X| = \sum_{k=1}^x |X_k|$ . We then define the welfare of an allocation region  $A$  as  $w(A) = \sum_{j \in [m]} \sum_{k \in [n]} (n - k + 1) \cdot |[\mu_j^{k-1}, \mu_j^k] \cap A_j|$ , where we set  $\mu_0^j = 0$  for all  $j \in [m]$ . Note that for a given allocation  $a$  and its corresponding allocation region  $A$ , we indeed have

$$\begin{aligned} w(A) &= \sum_{j \in [m]} \sum_{k \in [n]} (n - k + 1) \cdot |[\mu_j^{k-1}, \mu_j^k] \cap A_j| = \sum_{j \in [m]} \sum_{k \in [n]} |[0, \mu_j^k] \cap A_j| \\ &= \sum_{j \in [m]} \sum_{k \in [n]} \min(\mu_j^k, a_j) = \sum_{i \in [n]} \sum_{j \in [m]} \min(p_{i,j}, a_j) = \sum_{i \in [n]} u_i(a) = w(a). \end{aligned}$$

*Proof of Theorem 5.* Let  $n, m$  be fixed,  $P \in \mathcal{P}_{n,m}$  be a profile,  $a^{\text{UD}}$  be the output of UTILDECOMP, and  $a^{\text{GD}}$  the output of GREEDYDECOMP for  $P$ . Let  $c_i^{\text{UD}}$  for  $i \in [n]$  be the voter contributions certifying the decomposability of  $a^{\text{UD}}$ ; i.e.,  $\sum_{i \in [n]} c_i^{\text{UD}} = a_j^{\text{UD}}$  for each  $j \in [m]$  and  $a_j^{\text{UD}} \leq p_{i,j}$  whenever  $c_{ij}^{\text{UD}} > 0$ .

Now, let  $A^{\text{UD}}$  and  $A^{\text{GD}}$  be the allocation regions corresponding to  $a^{\text{UD}}$  and  $a^{\text{GD}}$ , respectively. We assign each voter  $i \in [n]$  the allocation region  $C_i^{\text{UD}} = [\sum_{i' \in [i-1]} c_{i'}^{\text{UD}}, \sum_{i' \in [i]} c_{i'}^{\text{UD}})$ . Intuitively, this

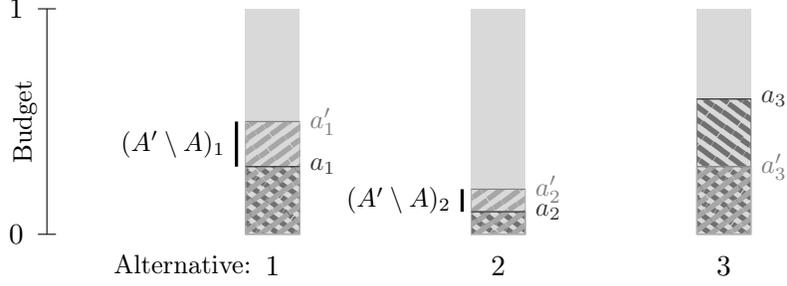


Figure 10: Illustration of example allocation regions. The exclusion of allocation regions  $A$  (dark gray, downwards hatching) and  $A'$  (light gray, upwards hatching) corresponding to the allocations  $a$  and  $a'$ , respectively, is given as  $(A' \setminus A) = ([a_1, a'_1], [a_2, a'_2], \emptyset)$ .

corresponds to letting the voters add their contribution to the allocation  $a^{\text{UD}}$  in order. We also construct the allocation regions  $C_i^{\text{GD}}$  for  $i \in [n]$  by tracking the order in which the voters spend their budget during the execution of GREEDYDECOMP. We initialize all voters regions as  $C_i^{\text{GD}} = (\emptyset)_{j \in [m]}$ . Whenever a voter's payment gets deducted from their budget in Line 9 of Algorithm 1, we add the interval  $[\tilde{a}_j, \tilde{a}_j + \pi_{i,j}(\tau^*)]$  to  $C_{ij}^{\text{GD}}$ .<sup>6</sup>

We decompose  $A^{\text{UD}}$  into two disjoint allocation regions  $A^{\text{UD}} \setminus A^{\text{GD}}$  and  $A^{\text{UD}} \cap A^{\text{GD}}$  and upper bound the additional welfare  $w(A^{\text{UD}} \setminus A^{\text{GD}})$  achieved by UTILDECOMP over GREEDYDECOMP. To this end, we further decompose  $A^{\text{UD}} \setminus A^{\text{GD}}$  by considering each voter's contribution to it, computing its welfare as  $w(A^{\text{UD}} \setminus A^{\text{GD}}) = \sum_{i \in [n]} w((A^{\text{UD}} \setminus A^{\text{GD}}) \cap C_i^{\text{UD}})$ . It is easy to see that for each alternative  $j$  the sets  $C_{ij}^{\text{GD}}$  of all voters form a disjoint partition of  $A_j^{\text{GD}} = [0, a_j^{\text{GD}})$  and that the contributions  $c_{ij}^{\text{GD}} = |C_{ij}^{\text{GD}}|$  certify the decomposability of  $a^{\text{GD}}$  (see proof of Proposition 3).

We will argue that, for each voter, their contribution to the welfare of  $A^{\text{UD}} \setminus A^{\text{GD}}$  is at most as big as their contribution  $w(C_i^{\text{GD}})$  to the welfare of  $A^{\text{GD}}$ . To see why, we first fix a voter  $i$ . If  $(A^{\text{UD}} \setminus A^{\text{GD}}) \cap C_i^{\text{UD}} = (\emptyset)_{j \in [m]}$ , then voter  $i$  does not contribute to  $A^{\text{UD}} \setminus A^{\text{GD}}$  and thus  $w((A^{\text{UD}} \setminus A^{\text{GD}}) \cap C_i^{\text{UD}}) = 0 \leq w(C_i^{\text{GD}})$  trivially holds. Otherwise, let  $k \in [n]$  be the minimum value such that  $[\mu_j^{k-1}, \mu_j^k] \cap (A_j^{\text{UD}} \setminus A_j^{\text{GD}}) \cap C_{ij}^{\text{UD}}$  is non-empty for some alternative  $j$ . Then, the welfare contribution of that voter to  $A_j^{\text{UD}} \setminus A_j^{\text{GD}}$  is upper-bounded by

$$\begin{aligned}
w((A^{\text{UD}} \setminus A^{\text{GD}}) \cap C_i^{\text{UD}}) &= \sum_{j \in [m]} \sum_{k'=k}^n (n - k' + 1) \cdot |[\mu_j^{k'-1}, \mu_j^{k'}] \cap (A_j^{\text{UD}} \setminus A_j^{\text{GD}}) \cap C_{ij}^{\text{UD}}| \\
&\leq \sum_{j \in [m]} \sum_{k'=k}^n (n - k' + 1) \cdot |[\mu_j^{k'-1}, \mu_j^{k'}] \cap C_{ij}^{\text{UD}}| \\
&\leq \sum_{j \in [m]} (n - k + 1) \cdot |[\mu_j^{k-1}, 1] \cap C_{ij}^{\text{UD}}| \\
&\leq \sum_{j \in [m]} (n - k + 1) \cdot |C_{ij}^{\text{UD}}| = \frac{n - k + 1}{n}.
\end{aligned}$$

Since  $[\mu_j^{k-1}, \mu_j^k] \cap (A_j^{\text{UD}} \setminus A_j^{\text{GD}})$  is non-empty, we know that GREEDYDECOMP spends less than  $\mu_j^k$  on alternative  $j$ . Now, consider the  $k^{\text{th}}$  outer iteration of GREEDYDECOMP after the last inner

<sup>6</sup>If  $\pi_{i,j}(\tau^*) = 0$  we do not add anything.

iteration with  $\tau^* = 1$ . At that time, we have  $\tilde{a}_j \leq a^{\text{GD}} < \mu_j^k$  and  $\tilde{a}_j \leq a^{\text{GD}} < a^{\text{UD}} \leq p_{i,j}$ . The only reason GREEDYDECOMP did not buy more of alternative  $j$  is that all voters above  $\tilde{a}_j$  have run out of budget. In particular, we thus know that voter  $i$  spent their whole budget on intervals below the  $k^{\text{th}}$  lowest vote, implying  $w(C_i^{\text{GD}}) \geq \frac{n-k+1}{n} \geq w((A^{\text{UD}} \setminus A^{\text{GD}}) \cap C_i^{\text{UD}})$ .

Before we conclude, we state two more observations that follow from the range respect of GREEDYDECOMP. First, the welfare from  $A^{\text{GD}} \setminus A^{\text{UD}}$  is at least  $|A^{\text{GD}} \setminus A^{\text{UD}}|$ . Second, since  $a_j^{\text{GD}} \geq \mu_j^1$  for all alternatives  $j$ , the welfare of  $A^{\text{UD}} \setminus A^{\text{GD}}$  is at most  $(n-1) \cdot |A^{\text{UD}} \setminus A^{\text{GD}}| = (n-1) \cdot |A^{\text{GD}} \setminus A^{\text{UD}}|$ . Thus,  $w(A^{\text{GD}} \setminus A^{\text{UD}}) \geq \frac{1}{n-1} w(A^{\text{UD}} \setminus A^{\text{GD}})$  and therefore

$$\begin{aligned}
w(a^{\text{UD}}) &= w(A^{\text{UD}} \cap A^{\text{GD}}) + w(A^{\text{UD}} \setminus A^{\text{GD}}) \\
&= w(A^{\text{GD}}) - w(A^{\text{GD}} \setminus A^{\text{UD}}) + w(A^{\text{UD}} \setminus A^{\text{GD}}) \\
&\leq w(A^{\text{GD}}) + \frac{n-2}{n-1} w(A^{\text{UD}} \setminus A^{\text{GD}}) \\
&= w(A^{\text{GD}}) + \frac{n-2}{n-1} \sum_{i \in [n]} w((A^{\text{UD}} \setminus A^{\text{GD}}) \cap C_i^{\text{UD}}) \\
&\leq w(A^{\text{GD}}) + \frac{n-2}{n-1} \sum_{i \in [n]} w(C_i^{\text{GD}}) \\
&\leq \left(2 - \frac{1}{n-1}\right) \cdot w(a^{\text{GD}}). \quad \square
\end{aligned}$$

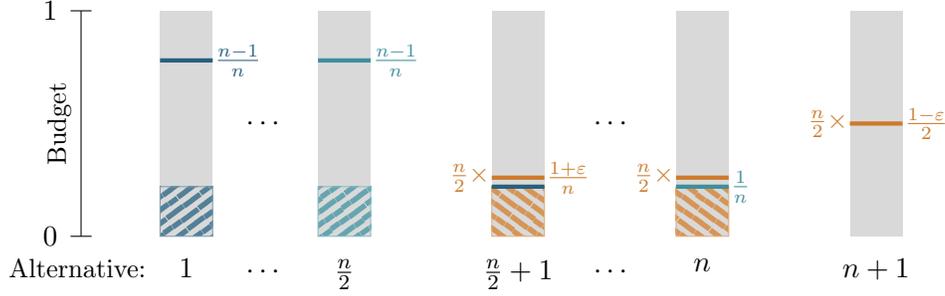
## D.5 Lower Bound on Welfare Ratio Between GREEDYDECOMP and UTILDECOMP

**Proposition 11.** *For every  $n \in \mathbb{N}$  with  $n \geq 4$  even, there are profiles with  $n$  voters in which UTILDECOMP achieves  $2 - \frac{6}{n+4}$  times the welfare that GREEDYDECOMP achieves.*

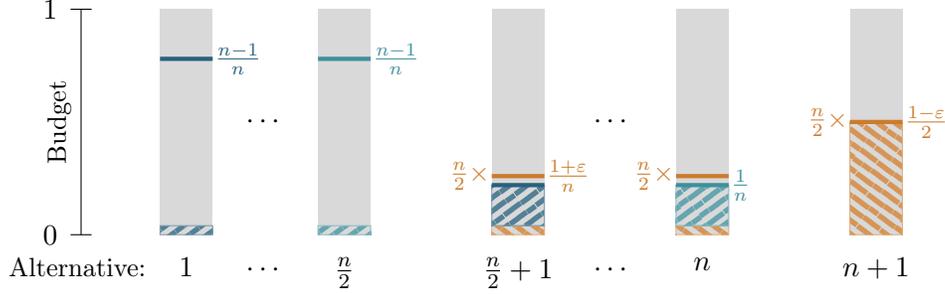
*Proof.* Let  $n \geq 4$  be an even number and  $\varepsilon > 0$ . Consider a profile with  $n$  voters and  $m = n + 1$  alternatives. Voters  $i$  from 1 to  $\frac{n}{2}$  vote  $\frac{n-1}{n}$  on alternative  $i$  and  $\frac{1}{n}$  on alternative  $i + \frac{n}{2}$ . All remaining voters vote  $\frac{1+\varepsilon}{n}$  on alternatives  $\frac{n}{2} + 1$  to  $n$  and  $\frac{1-\varepsilon}{2}$  on alternative  $n + 1$ . Figure 11 illustrates this profile.

GREEDYDECOMP first distributes the entire budget from the second half of the voters over alternatives  $\frac{n}{2} + 1$  to  $n$  and then has to spend the budget of each remaining voter  $i$  fully on alternative  $i$  (compare Figure 11a), leading to the allocation  $a = (\frac{1}{n}, \dots, \frac{1}{n}, 0)$  with welfare  $w(a) = \frac{n}{2} \cdot \frac{1}{n} + \frac{n}{2} \cdot (\frac{n}{2} + 1) \cdot \frac{1}{n} = \frac{n}{4} + 1$ .

A better decomposable allocation is  $a' = (\frac{\varepsilon}{n}, \dots, \frac{\varepsilon}{n}, \frac{1}{n}, \dots, \frac{1}{n}, \frac{1-\varepsilon}{2})$  with welfare  $w(a') = \frac{n}{2} \cdot \frac{\varepsilon}{n} + \frac{n}{2} \cdot (\frac{n}{2} + 1) \cdot \frac{1}{n} + \frac{n}{2} \cdot \frac{1-\varepsilon}{2} = \frac{n}{2} + \frac{1}{2} - \varepsilon \cdot (\frac{n}{4} - \frac{1}{2})$ . To see that  $a'$  is decomposable, consider the voter



(a) Output of GREEDYDECOMP.



(b) Welfare-optimal decomposable allocation.

Figure 11: Visualization of the profile and aggregates from the proof of Proposition 11.

contributions (compare Figure 11b)  $c \in [0, 1]^{n \times m}$  given by

$$c_{i,j} = \begin{cases} \frac{\varepsilon}{n} & \text{if } j = i, \\ \frac{1-\varepsilon}{n} & \text{if } j = i + \frac{n}{2}, \\ 0 & \text{otherwise,} \end{cases} \quad \text{for } i \in [\frac{n}{2}], j \in [m],$$

$$c_{i,j} = \begin{cases} \frac{\varepsilon}{n} & \text{if } j = i, \\ \frac{1-\varepsilon}{n} & \text{if } j = n + 1, \\ 0 & \text{otherwise,} \end{cases} \quad \text{for } i \in \{\frac{n}{2} + 1, \dots, n\}, j \in [m].$$

Thus, GREEDYDECOMP achieves a welfare approximation with respect to UTILDECOMP no better than

$$\frac{\frac{n}{2} + \frac{1}{2} - \varepsilon \cdot \left(\frac{n}{4} - \frac{1}{2}\right)}{\frac{n}{4} + 1} \xrightarrow{\varepsilon \rightarrow 0} \frac{\frac{n}{2} + \frac{1}{2}}{\frac{n}{4} + 1} = 2 - \frac{6}{n+4}. \quad \square$$

## E The Weighted Setting

In the weighted setting, each voter  $i \in [n]$  has a weight  $\omega_i \in \mathbb{N}$ , which intuitively indicates the voter's influence over the outcome. In the donor coordination interpretation (compare Brandl et al. [2022], Brandt et al. [2023]), the weights represent voters' contributions to a shared budget, which is then allocated across a set of charities. We denote the total weight (i.e., the total available budget) as  $b = \sum_{i \in [n]} \omega_i$ . In the remainder of this section, we introduce weighted versions of our mechanisms and axioms and show that all results from this paper extend to the weighted setting.

## E.1 Weighted Mechanisms

Every mechanism from the unweighted setting can be transformed into a weighted mechanism with the following simple construction. Given a weighted profile  $P$  with  $n$  voters and weights  $\omega$ , we construct the unweighted profile  $P'$  with  $n' = b = \sum_{i \in [n]} \omega_i$  voters by duplicating each voter  $i$  exactly  $\omega_i$  times.<sup>7</sup> We then run the unweighted mechanism on  $P'$  and use the resulting allocation as the output for the weighted profile  $P$ . For this section, let  $\mathbb{P}(P, \omega)$  denote the unweighted profile constructed from a weighted profile  $P$  with weights  $\omega$ .

We introduce a generalization of the class of moving-phantom mechanism to the weighted setting, which we then show to be equivalent to the class obtained by applying moving-phantom mechanisms as explained above. This generalization is based on the following key observation. An unweighted profile  $\mathbb{P}(P, \omega)$  constructed from a weighted profile potentially has a lot of duplicate voters. When computing the median of these voters and some moving phantoms on an alternative  $j$ , many of these phantoms can never be the median (unless they coincide with other phantoms or voters). As an example, consider a profile with one voter voting 0.2 and two voters voting 0.4 on some alternative  $j$ . On this alternative, there exists no placement of four phantoms  $f_0$  to  $f_3$ , such that  $f_2$  is the median (without being at the same position as another phantom or voter). By removing such phantoms from consideration, we can define weighted moving-phantom mechanisms as follows.

**Weighted Moving Phantoms** A weighted moving-phantom mechanism is defined via one phantom system for each value of the total budget  $b$ . For  $b \in \mathbb{N}$ , let  $\mathcal{F}_b = \{f_\kappa : \kappa \in [b]_0\}$  be a phantom system. Then, a family of phantom systems  $\mathcal{F} = \{\mathcal{F}_b \mid b \in \mathbb{N}\}$  defines the weighted moving-phantom mechanism  $\mathcal{B}^{\mathcal{F}}$  as follows. For a given profile  $P = (p_1, \dots, p_n)$  with  $n$  voters and weights  $\omega = (\omega_1, \dots, \omega_n)$  totaling  $\sum_{i \in [n]} \omega_i = b$ , let  $t^* \in [0, 1]$  be a value for which

$$\sum_{j \in [m]} \text{med}(f_{\Omega_{0,j}}(t^*), f_{\Omega_{1,j}}(t^*), \dots, f_{\Omega_{n,j}}(t^*), p_{1,j}, \dots, p_{n,j}) = 1,$$

where  $\Omega_{i,j}$  is the total weight of voters with the first to  $i^{\text{th}}$  lowest votes on alternative  $j$  (with some arbitrary tie-breaking). The weighted moving-phantom mechanism  $\mathcal{B}^{\mathcal{F}}$  returns the allocation  $a$  with  $a_j = \text{med}(f_{\Omega_{0,j}}(t^*), f_{\Omega_{1,j}}(t^*), \dots, f_{\Omega_{n,j}}(t^*), p_{1,j}, \dots, p_{n,j})$  for each alternative  $j$ .

We can show that running a weighted phantom mechanism induced by a family of phantom systems  $\mathcal{F}$  on some profile  $P$  with weights  $\omega$  is indeed the same as running the unweighted phantom mechanism induced by  $\mathcal{F}$  on the unweighted profile  $\mathbb{P}(P, \omega)$ . Note that, although the outcomes are identical, using weighted phantom mechanisms reduces the computation time from pseudo-polynomial to polynomial.

**Proposition 12.** *For any weighted profile  $P$  with weights  $\omega$ , the weighted phantom mechanism  $\mathcal{B}^{\mathcal{F}}$  induced by a family of phantom systems  $\mathcal{F}$  returns the output of the unweighted moving-phantom mechanism  $\mathcal{A}^{\mathcal{F}}$  for the profile  $\mathbb{P}(P, \omega)$ , that is,  $\mathcal{B}^{\mathcal{F}}(P, \omega) = \mathcal{A}^{\mathcal{F}}(\mathbb{P}(P, \omega))$ .*

*Proof.* Let  $P$  be a weighted profile with weight vector  $\omega$  and  $n$  voters. Let  $P' = \mathbb{P}(P, \omega)$  be the corresponding unweighted profile with  $b = \sum_{i \in [n]} \omega_i$  voters. Now, let  $\mathcal{F}_b = \{f_k \mid k = 0, \dots, b\}$  be the phantom system inducing  $\mathcal{B}^{\mathcal{F}}$ .

<sup>7</sup>Becker et al. [2025] use the same construction to lift their results for the computational complexity of computing Nash equilibria into the weighted setting. In our model, their results imply the existence and a pseudo-polynomial time computation of decomposable allocations in the weighted setting.

We fix a time  $t \in [0, 1]$  and an alternative  $j \in [m]$ . Let  $x = \text{med}(f_0(t), \dots, f_b(t), p'_{1,j}, \dots, p'_{b,j})$  be the median value of the votes from  $P'$  and the phantoms and let  $k'$  be the number of voters that vote strictly below  $x$  on alternative  $j$  in  $P'$ . Since  $x$  is the median and there are  $b - k'$  voters voting at least  $x$  on  $j$ , there must be at least  $k' + 1$  phantoms on positions at least  $x$  at time  $t$ , so in particular  $f'_{k'}(t) \geq x$ . Then, by the definition of  $P'$ , there must be an index  $k \in [n]$  with  $k' = \Omega_{k,j}$ . Then, there are exactly  $k$  voters voting strictly below  $x$  in  $P$  and thus  $n - k$  voters voting at least  $x$  on  $j$ . Additionally, we have the  $k + 1$  phantoms  $f_{\Omega_0,j}(t) \geq \dots \geq f_{\Omega_k,j}(t) = f'_{\Omega_k,j}(t) = f'_{k'}(t) \geq x$  voting at least  $x$  in  $P$ . We can analogously show that the number of voters and phantoms voting below or equal  $x$  in  $P$  is at least  $n + 1$ , showing that  $x$  is indeed also the median of the phantoms  $f_{\Omega_0,j}$  to  $f_{\Omega_n,j}$  and votes in  $P$  at time  $t$ .

Since at any time  $t$ , the medians of both mechanisms are the same, they will reach normalization at the same time and return the same allocation.  $\square$

We define the weighted version of GREEDYDECOMP as the mechanism that first converts the weighted profile into an unweighted one and then applies GREEDYDECOMP. This algorithm can also be formulated independently of the unweighted version, which would involve assigning initial budgets proportional to voters' weights, running the outer loop for  $k \in 1, \dots, b$  and charging voters in  $N_j^+$  proportionally to their weights. Such a reformulation, however, would require redefining the values  $\mu_j^k$ , and since it does not yield many additional insights, we omit it.

Both the weighted GREEDYDECOMP mechanism and the weighted moving-phantom mechanisms admit a further generalization to real-valued weights  $\omega_i$ . A trivial adaptation would lead to an infinite number of outer iterations for GREEDYDECOMP and an infinite number of phantoms for moving-phantom mechanisms. However, only a polynomial number of iterations/phantoms are actually relevant, and we can compute the outcomes by carefully controlling which outer iterations are executed in GREEDYDECOMP or, similarly, selecting the appropriate phantoms to consider in the moving-phantom mechanism.

## E.2 Weighted Axioms

We identified an easy way of constructing weighted mechanisms from unweighted ones. However, it is unclear whether the properties of these mechanisms carry over into the weighted setting. In this section, we show that this is indeed the case for the axioms considered in this paper. Note that this is not generally true, for instance, the axiom *individual fair share (IFS)* [Bogomolnaia et al., 2005, Brandl et al., 2021], which requires each voter's utility to be at least  $\frac{1}{n}$ , naturally generalizes requiring that each voter's utility is at least  $\frac{\omega_i}{b}$  in the weighted setting. Weighted IFS, however, is not necessarily satisfied when applying a mechanism satisfying IFS to the unweighted equivalent of a weighted profile.

Since a major strength of moving-phantom mechanisms is truthfulness, we start by investigating whether truthfulness still holds in the weighted setting. To show that weighted moving-phantom mechanisms are indeed truthful, we observe via standard arguments that a slightly stronger property holds: No group of voters with the same preferences can profit from a manipulation, even if we allow these voters to have different misreports.

We say that a budget-aggregation mechanism  $\mathcal{A}$  is *unanimous-group truthful* if for all  $n, m \in \mathbb{N}$  with  $m \geq 2$ , profile  $P = (p_1, \dots, p_n) \in \mathcal{P}_{n,m}$ , voter group  $N \subset [n]$  with  $p_i = p_h$  for all  $i, h \in N$ , and misreports  $p'_i \in \Delta^{(m-1)}$  for each  $i \in N$ , we have  $u_i(\mathcal{A}(P)) \geq u_i(\mathcal{A}(P'))$  for all  $i \in N$  where  $P'$  is identical to  $P$ , except that each voter  $i \in N$  votes  $p'_i$ .

**Proposition 13.** *Every truthful mechanism is unanimous-group truthful.*

*Proof.* Suppose there is a truthful, but not unanimous-group truthful mechanism  $\mathcal{A}$ . Then there must be  $n, m \in \mathbb{N}$ , a voter group  $N$ , a profile  $P$ , misreports  $p'_i$  for  $i \in N$  and the resulting profile  $P'$  as in the definition of unanimous-group truthfulness, such that  $u_i(\mathcal{A}(P)) < u_i(\mathcal{A}(P'))$  for some  $i \in N$ .

We fix an arbitrary order over  $N$  and consider the profiles  $P^k$  for  $k \in [|N|]_0$  in which only the first  $k$  voters from  $N$  misreport. Therefore, since  $u_i(\mathcal{A}(P^0)) = u_i(\mathcal{A}(P)) < u_i(\mathcal{A}(P')) = u_i(\mathcal{A}(P^{|N|}))$ , there must be some index  $k^* \in [|N|]$ , where  $u_i(\mathcal{A}(P^{k^*-1})) < u_i(\mathcal{A}(P^{k^*}))$ . Since all voters in  $N$  have the same utility function as  $i$ , this is also an improvement for the  $k^*$ th voter in  $N$ , which contradicts the truthfulness of  $\mathcal{A}$ .  $\square$

Because each weighted voter corresponds to a unanimous group in the unweighted profile, any truthfulness violation in the weighted setting carries over to a unanimous-group truthfulness violation in the unweighted setting. We therefore obtain the following corollary.

**Corollary 4.** *Weighted moving-phantom mechanisms are strategy-proof.*

While the truthfulness axiom did not require adaptation to the weighted setting, single-minded proportionality and decomposability require slight modifications. For single-minded proportionality, we require the output of a mechanism to coincide with the *weighted* mean of the votes in single-minded instances. It is straightforward to verify that any mechanism satisfying single-minded proportionality in the unweighted setting also satisfies it in the weighted setting. The definition of decomposability is modified only in that the contributions of each voter  $i$  sum up to  $\frac{w_i}{b}$  instead of  $\frac{1}{n}$ . This reflects the intuition that, in a donor coordination setting, each participant should be satisfied with how their contribution is spent. Again, it is easy to verify that every mechanism satisfying decomposability in the unweighted setting also does so in the weighted setting.

Since all relevant weighted axioms and mechanisms reduce to their unweighted counterparts when all voter weights are 1, the incompatibility of neutrality, decomposability, and truthfulness established in Proposition 1 remains valid in the weighted setting. The same holds for the fact that GREEDYDECOMP and UTILDECOMP are not Pareto efficient, stated in Proposition 9.

### E.3 Social Welfare in Weighted Budget Aggregation

We now discuss how the results in this paper can be lifted to the weighted setting. We start by defining the social welfare of an allocation  $a$  for a weighted instance as  $w(a) = \sum_{i \in [n]} w_i u_i(a)$ . Each voter's effect on the social welfare is scaled by its weight, which matches the definition of utilitarian welfare in Brandt et al. [2023]. For any aggregate, its social welfare in a weighted instance equals its social welfare in the corresponding unweighted instance. Therefore, all social welfare bounds established in Section 3 extend to the weighted setting, with every dependence on the number of voters replaced by a dependence on the total voter weight  $b$ . In particular, the lower bound on the welfare approximation of proportional phantoms from Proposition 5 and the matching upper bounds for UTILPROP (Corollary 1) and GREEDYDECOMP (Corollary 2) hold in the weighted setting with respect to the total weight of the voters  $b$ . We leave as an open question how these bounds change when expressed in terms of  $n$  in the weighted setting. We remark that the bounds with respect to  $m$  remain unchanged for the weighted setting.

For the welfare domination relation  $\succeq$  defined in Section 4, all our results carry over into the weighted setting without modification. In particular, Lemma 4 and the resulting Theorem 3 remain valid for the weighted moving-phantom mechanisms.

In section Section 5 we show that computing a welfare-optimal decomposable allocation is NP-hard, which remains true in the weighted setting, as it is more general. Finally, weighted GREEDYDECOMP provides an approximation to the welfare-optimal decomposable allocation, just as in the unweighted case (Theorem 5), with the dependence on the number of voters replaced by a dependence on  $b$ .